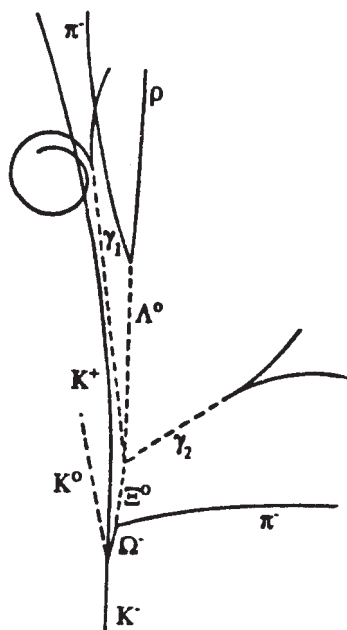


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**CLASSICAL AND QUANTUM SYMMETRIC
ELECTROMAGNETISM I: BASIC EQUATIONS****Micho Durdevich**

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Abstract

We study algebraic and geometric properties of generalized electromagnetic field configurations. Our basic principle is that electric and magnetic fields are mutually symmetric and interchangeable, much in the spirit of the original Dirac considerations. This leads us to naturally modify Maxwell equations, to make such a symmetry explicit. A subtle refinement of the symmetry concept is required, to adopt the apparent geometrical difference between electric field vectors and magnetic field pseudovectors. The symmetry between two fields leads to a gauge theory setting. The corresponding connection field is introduced and analyzed. The geometrical framework naturally invites to consider an extended space of internal degrees of freedom, by complementing the electromagnetic field with a scalar field, which generates the charge and current densities. The gauge-invariant Maxwell equations are discussed in detail. In particular, the energy and momentum conservation laws are derived. The theory will gradually be developed, and it will include the appropriate quantum space-time structures. The present paper deals with classical special-relativistic space-time only. Illustrative examples are presented.

1 Introduction

In this mini-series of papers, we shall present a complete formulation of a generalized electrodynamics on quantum and classical spaces, in a spirit of symmetry transformations and principal bundles. Our guiding idea is that electric and magnetic fields are objects of the same nature, and that consequently there should exist a completely localizable symmetry between them. We shall generalize the Maxwell equations accordingly.

This new symmetry, when localized as a gauge symmetry and viewed as a manifestation of a physical-geometrical connection field, possess very interesting properties. On one hand, it bears a strong resemblance with the classical Heaviside gravitational field [6, 8]. On the other, it provides a natural framework to consider stable objects with forms, and develop particle models entirely based on electromagnetism.

This unified framework for the symmetric electromagnetism will be considered at both classical geometrical, as well as the quantum-geometrical [2, 14, 17] levels.

The present study deals, basically, with the classical structures. We shall start with the standard Maxwell equations, and then carefully introduce and develop the symmetric geometrical framework. Our principal concern will be the study of generalized Maxwell equations, and computation of some illustrative examples.

The paper is organized in the following way. In the next section, we start from a modified system of Maxwell equations, so that electric and magnetic fields appear on completely equal footing. This naturally leads to the introduction of the $SO(2)$ symmetry group, formalizing the harmonic relation between these fields. We shall introduce a single complex field, as a complex combination of electric and magnetic fields. We next take into account that physically observed quantities never really distinguish between electricity and magnetism, which naturally leads us to consider a full associated gauge symmetry group operating on the electromagnetic field configurations. In order for this new symmetry to integrate consistently into the picture, we have to introduce a connection form, on the corresponding principal $SO(2)$ -bundle over the space-time. In such a way, we arrive to a gauge invariant form of Maxwell equations. We shall also derive a couple

of elementary consequences of these equations.

In our formulation, electromagnetic field configurations are interpretable as trace zero endomorphisms of a complex 2-dimensional vector bundle (over the space-time manifold). If we extend this, to consider all endomorphisms, then a ‘baby electromagnetic field’, emerges: It is just a ‘missing’ scalar part. This allows us to write the charge and current densities, as the covariant derivative components of the scalar field. We can imagine this field as an integral part of the complete electromagnetic field.

Next, properties of the basic fields—the connections and the extended electromagnetic field—will be studied. It turns out that, under certain invertibility assumptions, the connection field is completely determined by the electromagnetic field. As already mentioned, the connection field has a principal attribute of ‘keeping things together’, and this is also related to its possible interpretation as a kind of Heaviside co-gravitational field. The generalized Maxwell equations are non-linear, as the first partial derivatives are quadratic expressions in terms of the electromagnetic and connection fields. This naturally leads us to a new particle model, in which the entire particle is built out of electromagnetic and connection fields. Such configurations are energetically stable. Our theory aims to explain the nature of mass of matter (and its associated current) to be of a *purely electromagnetic origin* as envisioned long ago by Joseph John Thomson [16] and others [7].

In Section 3 some illustrative examples are given. In particular, we discuss the notion of stationary and spherically-symmetrical fields. These concepts are considered in the light of a full gauge-invariance of the presented theory. As particularly interesting solutions, we present a stable particle with a finite complete energy. We also present some non-standard transversal spherically symmetrical configurations. All these configurations are impossible within the framework of the standard electrodynamics.

In Section 4 some concluding remarks are made. We point out the relations with Mach principle, and ideas of Tesla. We outline how things will be further developed in the sequel to the present study.

Finally, in two appendices we have collected some basic facts and formulae from vector analysis and differential geometry. We refer to [9] as a classical treatise on differential geometry, a beautiful foundational approach rooted in principal bundles. A particular attention will be given to gauge transformations, and Lorentz and Poincare invariance of the theory.

2 Gauge Invariant Maxwell Equations

2.1 Complex Electromagnetic Field

To begin with, let us write down a modified version of Maxwell equations, allowing a possibility of the existence of magnetic monopoles. We refer to [10] as a classical treatise on standard electrodynamics and special and general relativity theory. Let $\mathbb{E}$ and $\mathbb{B}$ be the electric and magnetic fields, and ρ_E , ρ_B , ι_E and ι_B the electric and magnetic charge densities, and corresponding currents. The Maxwell equations are then

$$\begin{aligned} \operatorname{div}(\mathbb{E}) &= \rho_E & -\operatorname{rot}(\mathbb{E}) &= \frac{1}{c} \left\{ \iota_B + \frac{\partial}{\partial t} \mathbb{B} \right\} \\ \operatorname{div}(\mathbb{B}) &= \rho_B & \operatorname{rot}(\mathbb{B}) &= \frac{1}{c} \left\{ \iota_E + \frac{\partial}{\partial t} \mathbb{E} \right\}. \end{aligned} \quad (2.1)$$

In such a way, the equations look particularly elegant, with electric and magnetic fields appearing on a completely equal footing.

Remark 2.1. A highly plausible objection to such an approach to modify the original Maxwell equations, is that one could equally enthusiastically try to develop a biology of hypothetical mammals, possessing six legs instead of just four.*

Remark 2.2. In order to be able to establish a true symmetry between electric and magnetic fields, both fields should behave exactly in the same way, relative to all possible transformations. In other words, they should both be of *the same geometrical nature*. Here we shall assume that they are both *vector fields*. On the other hand, the rotor of a vector field is a pseudovector field. In order to have a theory symmetric under reflections of the physical space, one natural and straightforward assumption would be that electric and magnetic fields *interchange their roles*, in addition to being transformed in the standard vectorial way, under all possible reflections. This would keep both fields mutually totally symmetric, however a

*As explained so eloquently some time ago by Prof Božidar Milić, during one of his excellent lectures on Electrodynamics, at Faculty of Physics of Belgrade University, Serbia/Yugoslavia.

simple redefinition $(\mathbb{E}, \mathbb{B}) \rightsquigarrow (\mathbb{E} + \mathbb{B}, \mathbb{E} - \mathbb{B})$ would again produce a pair of fields satisfying the same equations, with the first field now vectorial and the second one pseudovectorial. Another possibility is to *combine* both fields into an irreducible complex vector field Φ , and to postulate a simple autonomous transformational property of the complex structure, corresponding to $i \leftrightarrow -i$. Such a symmetry is simply inexpressible in terms of real components $\mathbb{E}$ and $\mathbb{B}$.

Let us thus consider a complex field $\Phi = \mathbb{E} + i\mathbb{B}$, as well as complex charge density $\rho = \rho_E + i\rho_B$ and current $\iota = \iota_E + i\iota_B$. Now the system of 4 Maxwell equations becomes a pair of equations for new complex fields:

$$\nabla \cdot \Phi = \text{div}(\Phi) = \rho \quad \nabla \times \Phi = \text{rot}(\Phi) = \frac{i}{c} \left\{ \iota + \frac{\partial}{\partial t} \Phi \right\}. \quad (2.2)$$

We see that in such a way the equations exhibit a clear circular symmetry. They are invariant if we simultaneously multiply Φ , as well as ρ and ι by an arbitrary unitary complex number. These multiplications effectively mix and transform electric and magnetic field components of Φ , reflecting the initial motivation to have a theory in which electric and magnetic fields are explicitly mutually symmetric.

Our next conceptual step is completely in a gauge-theoretical spirit. We would like to *localize* the above mentioned $\text{SO}(2)$ -symmetry, and introduce *localized version* of the symmetry group. In other words the gauge group $\mathcal{G} = C^\infty(M, \text{SO}(2))$ is acting in the following way on the fields in the game:

$$\Phi \mapsto e^{if} \Phi \quad \rho \mapsto e^{if} \rho \quad \iota \mapsto e^{if} \iota \quad (2.3)$$

where $f: M \rightarrow \mathbb{R}$ is an arbitrary real smooth function over the space-time M . In order to make such a symmetry intrinsic, this requires an introduction of two additional fields: one vector field $\mathbb{A}$ and one scalar field φ , which are the space-time representation of a *connection form*

$$\omega \leftrightarrow (\varphi, \mathbb{A})$$

defined on a principal $\text{SO}(2)$ -bundle P over M . These fields transform as follows under the above symmetries:

$$\varphi \mapsto \varphi + \frac{1}{c} \frac{\partial}{\partial t} f \quad \mathbb{A} \mapsto \mathbb{A} + \nabla(f). \quad (2.4)$$

The gauge transformations from $\mathcal{G}$ are naturally interpreted as vertical automorphisms of the principal bundle P . In this geometrical setting, the gauge invariant Maxwell equations are

$$\nabla_\omega \cdot \Phi = \rho \quad \nabla_\omega \times \Phi = \frac{i}{c} \left\{ \iota + \frac{\partial_\omega}{\partial t} \Phi \right\} \quad (2.5)$$

where the differential operators assume their covariant version

$$\nabla_\omega = \nabla + i\mathbb{A} \quad \frac{\partial_\omega}{\partial t} = \frac{\partial}{\partial t} + ic\varphi \quad (2.6)$$

which has an intrinsic geometrical meaning, in terms of the covariant derivative components.

2.2 Invisible Scalar Companion

Our equations so far, however, do not tell us anything about objects ρ and ι . In fact *every possible* configuration of Φ uniquely determines ρ and ι so that the equations are satisfied. Therefore our system of equations is incomplete. There is a natural way to overcome this incompleteness, coming from geometrical considerations. As discussed in more detail in Appendix B, we can introduce a complex scalar field μ , which complements the complex 3-vector Φ in a natural decomposition $M_2(\mathbb{C}) = \mathbb{C} \oplus \mathfrak{sl}(2, \mathbb{C})$ into scalar matrices and those of the trace zero. This leads to the following refinements

$$\rho = -\frac{1}{c} \frac{\partial_\omega}{\partial t} \mu \quad \iota = c \nabla_\omega(\mu). \quad (2.7)$$

With these substitutions, the equations now read

$$\nabla_\omega \cdot \Phi + \frac{1}{c} \frac{\partial_\omega}{\partial t} \mu = 0 \quad \nabla_\omega \times \Phi = i \left\{ \nabla_\omega \mu + \frac{1}{c} \frac{\partial_\omega}{\partial t} \Phi \right\}. \quad (2.8)$$

By construction, everything is gauge-invariant and covariant, as it is expressed in terms of geometry of the spinorial structure of the Minkowski space-time.

Remark 2.3. We shall refer to (2.5) as well as (2.8) as general *complex Maxwell equations* or simply *Maxwell equations*. The connection fields $\mathbb{A}$

and φ have nothing to do with the vector and scalar electromagnetic potentials from the standard theory. In fact, due to the complete symmetry between electric and magnetic fields, it is not possible to introduce these electromagnetic potentials in our more general settings.

Remark 2.4. At a first sight, it might appear as an overkill to try expressing the electromagnetic 4-current as the covariant derivative of a single complex scalar field—the scalar field simply has not enough degrees of freedom to cover all varieties of configurations involving the electromagnetic charges. However, within the covariant derivative operator, a connection form implicitly appears, so in fact the equation (2.7) pretty much plays the role of a defining equation for the scalar and vector components φ and $\mathbb{A}$ of the connection.

Expressed in terms of the original electric and magnetic fields, the above system of equations reads equivalently

$$\begin{aligned} \text{rot}(\mathbb{E}) &= \mathbb{A} \times \mathbb{B} - \varphi \mathbb{E} - \mathbb{A} \epsilon - \text{grad}(\beta) - \frac{1}{c} \frac{\partial}{\partial t} \mathbb{B} \\ \text{div}(\mathbb{E}) &= -\frac{1}{c} \frac{\partial}{\partial t} \epsilon + \mathbb{A} \cdot \mathbb{B} - \varphi \beta \\ \text{div}(\mathbb{B}) &= -\frac{1}{c} \frac{\partial}{\partial t} \beta - \varphi \epsilon - \mathbb{A} \cdot \mathbb{E} \\ \text{rot}(\mathbb{B}) &= \text{grad}(\epsilon) - \mathbb{A} \beta - \mathbb{A} \times \mathbb{E} - \varphi \mathbb{B} + \frac{1}{c} \frac{\partial}{\partial t} \mathbb{E} \end{aligned} \tag{2.9}$$

where $\mu = \epsilon + i\beta$, scalar counterparts of electric and magnetic fields. The above formulae give explicit expressions for the charge and current densities, in terms of the connection, scalar and electromagnetic fields.

Remark 2.5. In geometrically intrinsic terms, we can say that Φ is a section of a complex vector bundle over the Minkowski space-time M . The theory is relativistically covariant, as discussed in Appendix B. When we speak of space and time coordinates, then we split the space-time M into a product of 3-dimensional physical space and one dimensional time continuum, both being Euclidean affine spaces. In this case we can also interpret configurations Φ as time-dependent sections of a complex vector bundle over the physical 3-space.

Remark 2.6. Our theory is completely, assuming that the connection field is defined globally on the whole of the space-time M : The velocity of change of entities Φ and μ is determined by their 3-space configurations in any given moment. This leads to a natural Hamiltonian formulation, and further dynamical refinements of the theory, which will be discussed later.

Under Lorentz transformations, the field Φ transforms in the following way:

$$\Phi_{||} \rightsquigarrow \Phi_{||} \quad \Phi_{\perp} \rightsquigarrow \frac{\Phi_{\perp} + i\mathbf{v} \times \Phi_{\perp}/c}{\sqrt{1 - v^2/c^2}}. \quad (2.10)$$

All this when viewed from a reference system in which the initial configuration moves with a uniform 3-velocity v .

The curvature of the connection ω is a gauge-invariant object, by construction. In standard space-time coordinates the curvature is given by the standard expression

$$F_{\mu\nu} = \partial_{\mu}\omega_{\nu} - \partial_{\nu}\omega_{\mu}$$

here $\omega_0 = \varphi$ and $(\omega_1, \omega_2, \omega_3) = \mathbb{A}$. If we now define

$$g = \frac{1}{c} \frac{\partial}{\partial t} \mathbb{A} - \nabla(\varphi) \quad \mathbb{K} = \text{rot}(\mathbb{A}) \quad (2.11)$$

the curvature reads

$$F \leftrightarrow \begin{pmatrix} 0 & g_1 & g_2 & g_3 \\ -g_1 & 0 & \mathbb{K}_3 & -\mathbb{K}_2 \\ -g_2 & -\mathbb{K}_3 & 0 & \mathbb{K}_1 \\ -g_3 & \mathbb{K}_2 & -\mathbb{K}_1 & 0 \end{pmatrix}. \quad (2.12)$$

We see that

$$\text{div}(\mathbb{K}) = 0 \quad \text{rot}(g) = \frac{1}{c} \frac{\partial}{\partial t} \mathbb{K}.$$

Let us now look at first elementary consequences of the complex Maxwell equations.

Proposition 2.1. *The following covariant continuity equation for the complex charge holds:*

$$\frac{\partial_{\omega}}{\partial t} \rho + \nabla_{\omega} \iota = c(\mathbb{K} + ig) \cdot \Phi. \quad (2.13)$$

Equivalently, we can write

$$\square_{\omega}(\mu) = c(\mathbb{K} + ig) \cdot \Phi \quad (2.14)$$

where $\square_{\omega} = \frac{\partial_{\omega}}{\partial t} \frac{\partial_{\omega}}{\partial t} - \nabla_{\omega}^2$ is the associated d'Alembert operator.

Proof. It follows by a direct calculation, using the above expressions for the curvature, as well as

$$\nabla_{\omega} \cdot (\nabla_{\omega} \times \Phi) = i \text{rot}(\mathbb{A}) \cdot \Phi \quad (2.15)$$

which holds for arbitrary complex vector field Φ . It is also worth noticing that

$$\nabla_{\omega} \times (\nabla_{\omega} \mu) = i \mu \text{rot}(\mathbb{A}) \quad (2.16)$$

where μ is an arbitrary complex scalar field. $\square$

Our next important step is to derive the conservation of energy equation for the electromagnetic field. We shall introduce the energy density and the Poynting vector, in the spirit of the standard formulation. So that

$$W = \frac{1}{2} \bar{\Phi} \cdot \Phi = \frac{1}{2} (\mathbb{E}^2 + \mathbb{B}^2) \quad \mathbb{P} = \frac{ic}{2} \Phi \times \bar{\Phi} = c \mathbb{E} \times \mathbb{B}. \quad (2.17)$$

Let us observe that both objects are completely invariant under the transformations from the gauge group $\mathcal{G}$.

Proposition 2.2. *For every configuration Φ satisfying the general Maxwell equations, we have*

$$\frac{\partial W}{\partial t} + \text{div}(\mathbb{P}) + \frac{1}{2} (\bar{\Phi} \cdot \iota + \bar{\iota} \cdot \Phi) = 0. \quad (2.18)$$

Proof. By a direct calculation :) We have

$$\text{div}(\bar{\Phi} \times \Phi) = \Phi \cdot (\overline{\nabla_{\omega} \times \Phi}) - (\nabla_{\omega} \times \Phi) \cdot \bar{\Phi} = -\frac{i}{c} (\bar{\Phi} \cdot \iota + \bar{\iota} \cdot \Phi) - \frac{i}{c} \frac{\partial}{\partial t} (\bar{\Phi} \cdot \Phi)$$

which easily assumes the form of the above continuity identity. $\square$

A physical interpretation of this formula, is that the energy of the electromagnetic field is transferred through the space, via the flux vector $\mathbb{P}$, and also a part of this energy is possibly transformed to a work on the charged particles participating in the game. The quantity

$$\lambda = \frac{1}{2}(\bar{\Phi} \cdot \iota + \bar{\iota} \cdot \Phi) \quad (2.19)$$

is thus interpretable as this work (per unit of time, per unit of volume). It is also a gauge-invariant quantity. It follows from the above formula, but it also follows directly from the continuity identity (2.18), and the gauge invariance of w and $\mathbb{P}$.

Remark 2.7. Such an interpretation is consistent, under the assumption that particles are objects existing ‘independently’ of the electromagnetic field. An entirely different situation arises when we postulate that all particles are built from the electromagnetic field. In this case it is reasonable to think that the electromagnetic field does not perform any dissipating energy work on itself!

Let us now define

$$\varpi = \frac{\bar{\mu}\mu}{2} \quad \vec{p} = \frac{c}{2}(\mu\bar{\Phi} + \bar{\mu}\Phi) \quad \vec{q} = \frac{c}{2i}(\mu\bar{\Phi} - \bar{\mu}\Phi) \quad (2.20)$$

as the appropriate scalar analogs of the energy density and Poynting vector, as well as the corresponding imaginary part.

The above expression for the work can be further simplified by substituting ι and using the first Maxwell equation. We have

$$\begin{aligned} \frac{1}{2}(\bar{\Phi} \cdot \iota + \bar{\iota} \cdot \Phi) &= \frac{c}{2}(\bar{\Phi} \cdot \nabla_{\omega}\mu + (\overline{\nabla_{\omega}\mu}) \cdot \Phi) = \frac{c}{2}\text{div}(\bar{\Phi}\mu + \Phi\bar{\mu}) - \\ &- \frac{c}{2}(\mu\overline{\nabla_{\omega} \cdot \Phi} + \bar{\mu}\nabla_{\omega} \cdot \Phi) = \text{div}(\vec{p}) + \frac{1}{2}(\mu(\overline{\frac{\partial_{\omega}}{\partial t}}\mu) + \bar{\mu}\frac{\partial_{\omega}}{\partial t}\mu) \\ &= \text{div}(\vec{p}) + \frac{1}{2}(\mu(\overline{\frac{\partial}{\partial t}}\mu) + \bar{\mu}\frac{\partial}{\partial t}\mu) = \text{div}(\vec{p}) + \frac{\partial}{\partial t}\varpi. \end{aligned}$$

By definition, these objects are gauge-invariant. Our continuity relation (2.18) becomes a very simple conservation

$$\frac{\partial}{\partial t}(W + \varpi) + \text{div}(\mathbb{P} + \vec{p}) = 0 \quad (2.21)$$

of the total energy of the vector and the scalar field.

Let us now study the conservation of the momentum of the electromagnetic field. In analogy with the standard theory, we define the Maxwell tensor (associated to the field configuration Φ) to be

$$\mathbb{M} = \begin{pmatrix} W & 0 & 0 \\ 0 & W & 0 \\ 0 & 0 & W \end{pmatrix} - \frac{1}{2}[\Phi \otimes \bar{\Phi} + \bar{\Phi} \otimes \Phi]. \quad (2.22)$$

Not surprisingly, this is also a gauge invariant object.

Proposition 2.3. *For every configuration Φ satisfying the general complex Maxwell equations, we have*

$$\frac{\partial}{\partial t} \mathbb{P} + \langle \nabla, \mathbb{M} \rangle + \frac{1}{2}(\bar{\rho} \cdot \Phi + \rho \cdot \bar{\Phi}) + \frac{i}{2c}(\iota \times \bar{\Phi} + \Phi \times \bar{\iota}) = 0. \quad (2.23)$$

Proof. Direct transformations, with the help of the first Maxwell equation, give

$$\begin{aligned} (\nabla_\omega \times \Phi) \times \bar{\Phi} - \Phi \times (\overline{\nabla_\omega \times \Phi}) &= (\bar{\Phi} \cdot \nabla_\omega) \Phi + (\Phi \cdot \bar{\nabla}_\omega) \bar{\Phi} - \nabla(\Phi \cdot \bar{\Phi}) \\ &= \langle \nabla, [\bar{\Phi} \otimes \Phi + \Phi \otimes \bar{\Phi}] \rangle - \bar{\rho} \cdot \Phi - \rho \cdot \bar{\Phi} - \nabla(\Phi \cdot \bar{\Phi}) \end{aligned}$$

and by the second Maxwell equation we have

$$\begin{aligned} \frac{\partial}{\partial t} \{\Phi \times \bar{\Phi}\} &= -ic(\nabla_\omega \times \Phi) \times \bar{\Phi} - \iota \times \bar{\Phi} + ic\Phi \times (\nabla_\omega \times \bar{\Phi}) - \Phi \times \bar{\iota} \\ &= \frac{c}{i} \langle \nabla, [\bar{\Phi} \otimes \Phi + \Phi \otimes \bar{\Phi}] \rangle - \frac{c}{i}(\rho \cdot \bar{\Phi} + \bar{\rho} \cdot \Phi) - \frac{c}{i} \nabla(\Phi \cdot \bar{\Phi}) - \iota \times \bar{\Phi} - \Phi \times \bar{\iota}. \end{aligned}$$

In terms of $\mathbb{P}$ and $\mathbb{M}$, it is precisely the desired identity. $\square$

The expression

$$\Lambda = \frac{1}{2}(\rho \cdot \bar{\Phi} + \bar{\rho} \cdot \Phi) + \frac{i}{2c}(\iota \times \bar{\Phi} + \Phi \times \bar{\iota}) \quad (2.24)$$

is interpretable as the *Lorentz force*. Clearly, it is gauge-invariant. Its gauge invariance also follows from the above continuity identity. A physical interpretation of this identity is that the local change of the field momentum is distributed to the momentum carried away by the field (the divergency of the Maxwell tensor) and the change of the momentum of the particles (the Lorentz force).

Lemma 2.4. *The Lorentz force is given by*

$$\Lambda = -\text{grad}(\varpi) - \frac{1}{c}\text{rot}(\vec{q}) - \frac{1}{c^2}\frac{\partial}{\partial t}\vec{p}. \quad (2.25)$$

Proof. A direct application of Maxwell equations gives

$$\begin{aligned} \frac{1}{2}(\rho \cdot \bar{\Phi} + \bar{\rho} \cdot \Phi) + \frac{i}{2c}(\iota \times \bar{\Phi} + \Phi \times \bar{\iota}) &= -\frac{1}{2c}\frac{\partial}{\partial t}(\mu\bar{\Phi} + \bar{\mu}\Phi) + \frac{1}{2c}\mu\overline{\left(\frac{\partial}{\partial t}\Phi\right)} \\ &+ \frac{1}{2c}\bar{\mu}\frac{\partial}{\partial t}\Phi + \frac{i}{2}[\text{rot}(\mu\bar{\Phi}) - \text{rot}(\bar{\mu}\Phi)] + \frac{i}{2}[\bar{\mu}(\nabla_\omega \times \Phi) - \mu(\bar{\nabla}_\omega \times \bar{\Phi})] \\ &= -\frac{1}{2c}\frac{\partial}{\partial t}(\mu\bar{\Phi} + \bar{\mu}\Phi) + \frac{i}{2}[\text{rot}(\mu\bar{\Phi}) - \text{rot}(\bar{\mu}\Phi)] - \frac{1}{2}\mu\bar{\nabla}_\omega(\bar{\mu}) - \frac{1}{2}\bar{\mu}\nabla_\omega(\mu) \\ &= -\frac{1}{c^2}\frac{\partial}{\partial t}\vec{p} - \frac{1}{c}\text{rot}(\vec{q}) - \text{grad}(\varpi). \end{aligned}$$

It is interesting to observe somehow counterintuitive appearance of minus sings. $\square$

Our conservation laws are expressible in a natural covariant form, with the help of the electromagnetic energy-momentum tensor:

$$\left(\frac{1}{c}\frac{\partial}{\partial t}, \nabla\right) \cdot \tau + (\lambda, \Lambda)^\top = 0 \quad \tau = \begin{pmatrix} w & \frac{1}{c}\mathbb{P}^\top \\ \frac{1}{c}\mathbb{P} & \mathbb{M} \end{pmatrix}. \quad (2.26)$$

By construction, this tensor is symmetric and gauge-invariant.

As explained in detail in Appendix B, there is a natural way to represent the field configuration (μ, Φ) as a complex 2×2 matrix

$$(\mu, \Phi) \leftrightarrow \begin{pmatrix} \mu + \Phi_3 & \Phi_1 - i\Phi_2 \\ \Phi_1 + i\Phi_2 & \mu - \Phi_3 \end{pmatrix}.$$

The determinant of this matrix is given by

$$\det(\mu, \Phi) = \mu^2 - \Phi \cdot \Phi.$$

By construction, it is invariant under Lorentz transformations. Maxwell equations can be written in the following symbolic and elegant spinorial form

$$\mathfrak{D}(\mu, \Phi) = 0 \quad (2.27)$$

where

$$\mathfrak{D} = \mathfrak{D} + i \begin{pmatrix} \varphi + \mathbb{A}_3 & \mathbb{A}_1 - i\mathbb{A}_2 \\ \mathbb{A}_1 + i\mathbb{A}_2 & \varphi - \mathbb{A}_3 \end{pmatrix} \quad \mathfrak{D} = \begin{pmatrix} \partial_0 + \partial_3 & \partial_1 - i\partial_2 \\ \partial_1 + i\partial_2 & \partial_0 - \partial_3 \end{pmatrix}. \quad (2.28)$$

As explained in detail in [13], the spinorial structure is encoded in the geometry of the light cones and light rays, of the Minkowski space-time. An important special case of electromagnetic configurations is when the matrix determinant is non-zero, which is equivalent to the invertibility of the configuration. In this case the inverse matrix is given by

$$(\mu, \Phi)^{-1} = \frac{1}{\det(\mu, \Phi)} \begin{pmatrix} \mu - \Phi_3 & -\Phi_1 + i\Phi_2 \\ -\Phi_1 - i\Phi_2 & \mu + \Phi_3 \end{pmatrix} = \frac{(\mu, -\Phi)}{\mu^2 - \Phi \cdot \Phi}. \quad (2.29)$$

Definition 2.8. Such configurations will be called regular configurations (over a given space-time region).

A very important property is that if (μ, Φ) satisfies Maxwell equations and is regular, then the connection field is completely determined by the configuration itself.

Proposition 2.5. For regular electromagnetic configurations, the scalar connection potential is given by

$$\varphi = \frac{i}{\mathbb{D}} \left\{ \frac{1}{2c} \frac{\partial}{\partial t}(\mathbb{D}) + \mu \operatorname{div}(\Phi) - \operatorname{grad}(\mu) \cdot \Phi - i \operatorname{rot}(\Phi) \cdot \Phi \right\} \quad (2.30)$$

while the vector potential is

$$\begin{aligned} \mathbb{A} = \frac{i}{\mathbb{D}} \left\{ \frac{1}{2} \operatorname{grad}(\mathbb{D}) + \frac{1}{c} \mu \frac{\partial}{\partial t}(\Phi) - \frac{1}{c} \frac{\partial}{\partial t}(\mu) \Phi + (\Phi \nabla) \Phi - \operatorname{div}(\Phi) \Phi \right\} + \\ + \frac{1}{\mathbb{D}} \left\{ \frac{1}{c} \frac{\partial}{\partial t}(\Phi) \times \Phi + \operatorname{grad}(\mu) \times \Phi - \mu \operatorname{rot}(\Phi) \right\} \end{aligned} \quad (2.31)$$

where $\mathbb{D} = \det(\mu, \Phi) = \mu^2 - \Phi \cdot \Phi$.

Proof. By separating the partial derivatives and connection components in (2.27) and multiplying by the inverse matrix $(\mu, \Phi)^{-1}$ we obtain

$$\begin{pmatrix} \varphi + \mathbb{A}_3 & \mathbb{A}_1 - i\mathbb{A}_2 \\ \mathbb{A}_1 + i\mathbb{A}_2 & \varphi - \mathbb{A}_3 \end{pmatrix} = i \left\{ \tilde{\partial}(\mu, \Phi) \right\} (\mu, \Phi)^{-1} \quad (2.32)$$

which turns out to be equivalent to (2.30) and (2.31). Indeed, by expressing everything in terms of Pauli matrices and performing elementary calculations we obtain a number of terms

$$\begin{aligned} \varphi + \mathbb{A} \cdot \vec{\sigma} &= \frac{i}{\mathbb{D}} \left\{ \frac{1}{c} \frac{\partial}{\partial t}(\mu) + \text{grad}(\mu) \cdot \vec{\sigma} + \text{div}(\Phi) + \frac{1}{c} \frac{\partial}{\partial t}(\Phi) \cdot \vec{\sigma} + i \text{rot}(\Phi) \cdot \vec{\sigma} \right\} (\mu, -\Phi) \\ &= \frac{i}{\mathbb{D}} \left\{ \frac{1}{c} \frac{\partial}{\partial t}(\mu) \mu + \mu \text{div}(\Phi) - \text{grad}(\mu) \cdot \Phi - \frac{1}{c} \frac{\partial}{\partial t}(\Phi) \cdot \Phi - i \text{rot}(\Phi) \cdot \Phi \right\} \\ &+ \frac{i\mu}{\mathbb{D}} \left\{ \frac{1}{c} \frac{\partial}{\partial t}(\Phi) + i \text{rot}(\Phi) + \text{grad}(\mu) \right\} \cdot \vec{\sigma} + \left\{ \frac{1}{c} \frac{\partial}{\partial t}(\Phi) + i \text{rot}(\Phi) + \text{grad}(\mu) \right\} \times \frac{\Phi}{\mathbb{D}} \cdot \vec{\sigma} \\ &\quad - \frac{i\Phi}{\mathbb{D}} \left\{ \frac{1}{c} \frac{\partial}{\partial t}(\mu) + \text{div}(\Phi) \right\} \vec{\sigma} = \frac{i}{\mathbb{D}} \left\{ \frac{1}{2c} \frac{\partial}{\partial t}(\mathbb{D}) \right\} + \\ &\quad + \frac{i}{\mathbb{D}} \left\{ \mu \text{div}(\Phi) - \text{grad}(\mu) \cdot \Phi - i \text{rot}(\Phi) \cdot \Phi \right\} \\ &\quad + \frac{i}{\mathbb{D}c} \left\{ \mu \frac{\partial}{\partial t}(\Phi) - \Phi \frac{\partial}{\partial t}(\mu) \right\} \cdot \vec{\sigma} + \frac{1}{\mathbb{D}} \left\{ \text{grad}(\mu) \times \Phi - \mu \text{rot}(\Phi) \right\} \cdot \vec{\sigma} \\ &+ \frac{1}{\mathbb{D}c} \left\{ \frac{\partial}{\partial t}(\Phi) \times \Phi \right\} \cdot \vec{\sigma} + \frac{i}{\mathbb{D}} \left\{ (\Phi \nabla) \Phi - \text{div}(\Phi) \Phi \right\} \cdot \vec{\sigma} + \frac{i}{\mathbb{D}} \left\{ \frac{1}{2} \text{grad}(\mu^2 - \Phi \cdot \Phi) \right\} \cdot \vec{\sigma} \end{aligned}$$

and the desired relations follow easily. $\square$

On the other hand, we can take an arbitrary configuration (μ, Φ) which is regular over a given region, and try to define the connection via (2.30) and (2.31). Clearly, the Maxwell equations will be formally satisfied, however in general such a connection will not be real. By requiring the reality of the connection components, we introduce an explicit condition on the configuration (μ, Φ) . Indeed according to the above formula for the connection, the conjugated version is

$$-i(\bar{\mu}, \bar{\Phi})^{-1} \left\{ (\bar{\mu}, \bar{\Phi}) \tilde{\partial} \right\} = \begin{pmatrix} \varphi + \mathbb{A}_3 & \mathbb{A}_1 - i\mathbb{A}_2 \\ \mathbb{A}_1 + i\mathbb{A}_2 & \varphi - \mathbb{A}_3 \end{pmatrix} \quad (2.33)$$

and thus the reality condition is equivalent to

$$(\mu, \Phi)^* \overleftarrow{\mathfrak{D}} (\mu, \Phi) + (\mu, \Phi)^* \overrightarrow{\mathfrak{D}} (\mu, \Phi) = 0. \quad (2.34)$$

Such a property for a regular configuration (μ, Φ) ensures that (2.32) is indeed a (unique possible) connection form complementing (μ, Φ) to an electromagnetic system.

Definition 2.9. For a general configuration (μ, Φ) we shall say that it is balanced (over a given space-time region) iff (2.34) holds.

It is worth observing that in the property of balanceness we can replace $\mathfrak{D}$ by an arbitrary covariant derivative $\mathfrak{D}$. So clearly, as we already proved by direct calculations, every electromagnetic system is balanced.

Proposition 2.6. *A necessary and sufficient condition for a configuration to be balanced is given by*

$$\tau(\mathfrak{D}) = \sum_{\nu} \partial^{\nu} \tau_{\nu}^{\mu} = 0. \quad (2.35)$$

In other words, the associated energy-momentum tensor τ should satisfy the continuity equation.

Proof. As explained in Appendix B, the electromagnetic energy-momentum tensor τ is given by $\tau(\xi) = (\mu, \Phi)\xi(\mu, \Phi)^*/2$. The above expression (2.34) can be reformulated as the symbolic identity

$$\tau(\mathfrak{D}) = 0$$

where the multiplication between $\mathfrak{D}$ components and that of τ is the interpreted as the differentiation action of former to the latter. In other words, (2.35) holds. $\square$

In summary, regular and balanced configurations are those that can be, in a unique manner, complemented to an electromagnetic system.

We shall now study non-regular electromagnetic configurations. If the configuration is non-regular in some space-time region (that is, the quantity

$\mathbb{D}$ reaches the value zero somewhere in the region) then the above considered procedures can not apply. We shall call these configurations singular configurations. For singular configurations, in general there will exist multiple connections so that the Maxwell equations are fulfilled. In fact a given (μ, Φ) all such configurations form an infinite-dimensional real affine space. A trivial example is when the electromagnetic field identically vanishes, and so any connection is good. A non-trivial example is given by the standard electromagnetic field in vacuum, as discussed in the next section.

The singularity (in a given space-time point) means that the configuration is either zero or that the kernel of (μ, Φ) is one-dimensional. This is equivalent to one-dimensionality property of the image of (μ, Φ) . These spaces give us a first necessary condition for possibly singular electromagnetic configurations.

Lemma 2.7. *For every electromagnetic configuration (μ, Φ) we have*

$$\ker_x \{ \tilde{\partial}(\mu, \Phi) \} \supseteq \ker_x \{ (\mu, \Phi) \} \quad (2.36)$$

in every space-time point x of its region of definition.

Proof. This is a direct consequence of (2.27) expressing $\tilde{\partial}(\mu, \Phi)$ as $-i$ times the composition of the matrices for the connection and the electromagnetic field. $\square$

Obviously this condition becomes trivial for regular configurations. On the other hand, if (μ, Φ) vanishes at some point, then it follows that $\tilde{\partial}(\mu, \Phi)$ vanishes too. It is easy to see that (2.36) is a gauge-invariant condition.

Definition 2.10. We shall call (μ, Φ) semi-regular (over a given space-time region) if it satisfies (2.36).

Let us now assume that (μ, Φ) is such that for every $x \in U$ the kernel of $(\mu, \Phi)_x$ is one-dimensional. This is equivalent to the following *diadic* representation

$$(\mu, \Phi) = \rho u \otimes v^* \quad (2.37)$$

where $u, v: U \rightarrow \mathbb{C}^2$ are unit vector fields and ρ is a strictly positive scalar function on the region U .

It is easy to see that each of the fields u and v is determined up to a gauge transformation over U . Fixing one of them fixes another. We have

$$\rho^2 u \otimes u^* = (\mu, \Phi)(\mu, \Phi)^* \quad \rho^2 v \otimes v^* = (\mu, \Phi)^*(\mu, \Phi). \quad (2.38)$$

It is worth mentioning that the field ρ is uniquely determined by the configuration (μ, Φ) , as we can see by taking the trace of either one of the above expressions.

Remark 2.11. The objects u , v and ρ have not an intrinsic geometrical meaning in terms of the spinors, as we used the scalar product in $\mathbb{C}^2$ which is not $\text{SL}(2, \mathbb{C})$ -invariant.

If we consider the action of the connection matrix $(\varphi, \mathbb{A})$ on the configuration, we see that we have the following degree of freedom that does not affect the action:

$$(\varphi, \mathbb{A}) \rightsquigarrow (\varphi, \mathbb{A}) + \zeta(1 - u \otimes u^*) \quad (2.39)$$

but the principal question here is if there exist at least one connection so that our configuration satisfy the Maxwell equations.

Lemma 2.8. *In the case of semi-regular configurations (μ, Φ) of the described type, a magic connection will exist iff the configuration is balanced.*

Proof. If (μ, Φ) is semi-regular, then there exists a field $\xi: U \rightarrow \mathbb{C}^2$ such that

$$\bar{\partial}(\mu, \Phi) = \xi \otimes v^*.$$

Now using the representation (2.37) the balanceness condition reads

$$\langle u, \xi \rangle + \langle \xi, u \rangle = 0$$

in other words the scalar product between u and ξ should be imaginary. The Maxwell equations (2.27) become the following condition for the connection field:

$$\xi \otimes v^* = -i\rho(\varphi, \mathbb{A})(u \otimes v^*) \quad \Leftrightarrow \quad \xi = -i\rho(\varphi, \mathbb{A})u.$$

It is easy to see that we can always find a hermitian matrix $(\varphi, \mathbb{A})$ satisfying this mapping condition. $\square$

Proposition 2.9. *Let (μ, Φ) be an arbitrary electromagnetic configuration with a flat connection ω . Then for every scalar field f self-parallel relative to ω the configuration $(f + \mu, \Phi)$ is balanced. Therefore, if in addition $(f + \mu, \Phi)$ is regular then there exists a unique connection complementing it to an electromagnetic system.*

Proof. If the connection is flat, we can make $(\varphi, \mathbb{A})$ disappear by a gauge transformation. In this case the Maxwell equations assume the classical form $\mathfrak{D}(\mu, \Phi) = 0$ and f is just a constant factor. In this case $\mathfrak{D}(\mu + f, \Phi) = 0$ obviously, and so the new configuration is balanced (and in a trivial way). If the new configuration is regular then there exists a unique associated connection, which is the initial one. $\square$

Let us end this section on a high note with this trivial, but useful and presented in a gauge invariant setting, statement. It is worth remembering that we can always add a constant scalar field to classical configurations and make them regular.

3 Examples

3.1 Almost Trivial Configurations

Let us begin with solutions without any electromagnetic field at all. In other words, $\Phi = 0$ everywhere. In this case, Maxwell equations are equivalent to

$$\frac{\partial_\omega}{\partial t} \mu = 0 \quad \nabla_\omega \mu = 0 \quad (3.1)$$

in other words, μ must be covariantly constant. If the curvature of the connection vanishes identically, then μ is determined by its values over an arbitrary point, which are then propagated to the entire space-time via the parallel transport. On the other hand, if there exists a point in which the curvature is non-zero, then μ is identically zero, and in this case the connection can be arbitrary.

Another very simple class of special solutions is given by standard electromagnetic configurations in vacuum, without the scalar field μ , so:

$$\operatorname{div}(\Phi) = 0 \quad \operatorname{rot}(\Phi) = \frac{i}{c} \frac{\partial}{\partial t}(\Phi) \quad \mu = 0. \quad (3.2)$$

Maxwell equations now reduce to a purely algebraic problem for φ and $\mathbb{A}$, namely

$$\mathbb{A} \cdot \Phi = 0 \quad \mathbb{A} \times \Phi = i\varphi\Phi. \quad (3.3)$$

Assuming that both Φ and $\mathbb{A}$ are not vanishing (over some region), it follows that $\mathbb{A}$ and φ are of the same intensity, and $\mathbb{A}$ is orthogonal to Φ . On the other hand, Φ represents an *electromagnetic wave*, in the sense of $\Phi \cdot \Phi = 0$, in other words, the electric and magnetic components are mutually orthogonal and of the same intensity. This classifies completely such solutions.

Remark 3.1. In particular, a standard non-wave configuration in vacuum of Φ , requires both connection and the scalar field to either vanish, or be non-trivial.

3.2 Stationary Solutions

When we speak of stationary things, we always assume that a factorization of the space-time into a product of 3-dimensional space and 1-dimensional time continuum is given. In the spirit of gauge-invariance, it is natural to adopt the following definition of stationarity.

Definition 3.2. We call a field configuration *weakly stationary*, if it evolves in such a way so that it never changes its gauge-equivalence class over the physical 3-space.

This is our main concept of stationarity, and in what follows we shall refer to weakly stationary configuration as simply stationary. The stationarity property of the connection field, the curvature and gauge transformations will be understood in the standard sense, as independence of the time variable.

Lemma 3.1. *If a field configuration is stationary then over a cylindric space-time region which does not contain zeros of the configuration, there*

exists a global gauge transformation such that the transformed configuration does not depend on time. Such a gauge transformation is determined modulo the subgroup of stationary gauge transformations.

Proof. Certainly, a zero of a stationary field is stationary, in the sense of remaining a zero during the time evolution. So in a region free of zeros, the gauge transformation moving one configuration into another is completely determined by these configurations. $\square$

Remark 3.3. A configuration might be stationary in a given region U of a physical 3-space which is everywhere dense, however not stationary as a configuration on the complete space-time.

The dependence on time variable t , and related to it the operator of partial derivative, is not a gauge invariant concept. Its intrinsic counterpart is the covariant derivative $\nabla_0 = c^{-1}\partial/\partial t + i\varphi$. So we can also naturally introduce another version of stationarity, by requiring that this covariant derivative vanishes.

Definition 3.4. A field configuration (μ, Φ) is called *strongly stationary* iff

$$\frac{1}{c}\frac{\partial}{\partial t}(\mu) + i\varphi\mu = 0 \quad \frac{1}{c}\frac{\partial}{\partial t}(\Phi) + i\varphi\Phi = 0. \quad (3.4)$$

Lemma 3.2. *Every strongly stationary configuration is weekly stationary. Moreover, there exists a gauge transformation such that φ vanishes and the field is independent on time variable.*

Proof. It is always possible to make φ disappear everywhere, by a suitable gauge transformation. It is clear that after such a transformation (3.4) just says that μ and Φ do not depend of time. $\square$

It is interesting to ask whether the stationarity of the configuration implies the stationarity of the connection field. It is not always the case. A trivial example is zero electromagnetic field with an arbitrary connection. However

Proposition 3.3. *If (μ, Φ) is a stationary solution of Maxwell equations, then we can always chose a connection which is independent on time. In particular, if (μ, Φ) is regular then its unique connection $(\varphi, \mathbb{A})$ does not depend on time.*

Proof. We can assume that (μ, Φ) does not depend on t . We can always take a snapshot of the connection in a given moment of time t and extend it by stationarity to the whole space-time region. Clearly such a new connection will be compatible with the configuration (μ, Φ) . In the case of regular solutions, we see that $(\varphi, \mathbb{A})$ by looking at the explicit expression (2.32). $\square$

Hence Maxwell's equations for a stationary field become

$$\begin{aligned} \operatorname{div}(\mathbb{E}) - \varphi\beta - \mathbb{A} \cdot \mathbb{B} &= 0 & \operatorname{rot}(\mathbb{E}) - \mathbb{A} \times \mathbb{B} &= -\operatorname{grad}(\beta) - \mathbb{A}\epsilon - \varphi\mathbb{E} \\ \operatorname{div}(\mathbb{B}) + \mathbb{A} \cdot \mathbb{E} + \varphi\epsilon &= 0 & \operatorname{rot}(\mathbb{B}) + \mathbb{A} \times \mathbb{E} &= -\varphi\mathbb{B} + \operatorname{grad}(\epsilon) - \mathbb{A}\beta. \end{aligned} \quad (3.5)$$

In a completely statical solution, restrictions of ρ on physical 3-space leafs will then be all mutually gauge equivalent. The curvature fields g and $\mathbb{K}$ will be static in the standard sense (that is, not depending on t).

A further special case of stationary solutions is given by *electrostatic* configurations. In this case Φ can be assumed to be real ($\Phi = \mathbb{E}$ and $\mathbb{B} = 0$) and our equations further simplify into

$$\begin{aligned} \operatorname{div}(\mathbb{E}) &= \beta\varphi & \mathbb{A} \cdot \mathbb{E} &= -\epsilon\varphi \\ \operatorname{grad}(\epsilon) &= \beta\mathbb{A} + \mathbb{A} \times \mathbb{E} & \operatorname{grad}(\beta) &= -\epsilon\mathbb{A} - \varphi\mathbb{E} - \operatorname{rot}(\mathbb{E}). \end{aligned} \quad (3.6)$$

3.3 Spherical Symmetricity

3.3.1 General Properties

We shall first describe all possible spherically symmetrical configurations Φ , without requiring that Maxwell equations be satisfied.

Proposition 3.4. *Let us consider a point of origin O and a configuration Φ which is spherically symmetrical, with a possible singularity at O . Then on any sphere S around O one of the following happens:*

- (i) *The section Φ vanishes on S ;*

- (ii) The restriction $\Phi|_S$ is proportional to the radial vector field e_r .
- (iii) For every point $u \in S$ the vector $\Phi(u)$ satisfies

$$R_{u,\phi}\Phi(u) = e^{i\phi}\Phi(u).$$

- (iv) For every point $u \in S$ the vector $\Phi(u)$ satisfies

$$R_{u,\phi}\Phi(u) = e^{-i\phi}\Phi(u).$$

Here $R_{u,\phi}$ is the orthogonal rotation around the oriented line spanned by u , by the angle ϕ .

Proof. In accordance to the general principles of gauge-invariance, this means that Φ has the property that after any orthogonal rotation around a given point of origin O , the configuration transforms into a configuration gauge equivalent to Φ . This implies that $\Phi(u)$ belongs to one of the 3 invariant one-dimensional subspaces for $R_{u,\phi}$ for every non-zero vector u , corresponding to eigenvalues 1, $e^{i\phi}$ and $e^{-i\phi}$ respectively. Due to the gauge-invariance of Φ the vector $\Phi(v)$ has the same transformation property relative to $R_{v,\phi}$, as does $\Phi(u)$ relative to $R_{u,\phi}$, for every v belonging to the sphere S centered at O and containing u . $\square$

Remark 3.5. Obviously if a spherically symmetrical Φ is defined at O then it vanishes at O . If Φ is of type (ii) on a sphere S then, because of the continuity of Φ , there will be an entire open shell of spheres containing S , such that Φ is of type (ii) on every sphere from the shell. Similarly, types (iii) and (iv) will be grouped into open shells. Between shells of different type there will always be at least one sphere on which Φ vanishes.

Definition 3.6. We shall say that a spherically-symmetrical Φ is of radial type, if it satisfies (ii) of the above proposition (on a given shell). The type (iii) will be called positive transversal, and similarly the type (iv) will be called negative transversal.

Remark 3.7. If Φ is radial on a given shell U , then we can perform a static gauge transformation so that the transformed Φ becomes real, in other words, with only the electric component $\mathbb{E} = \mathbb{E}(r)e_r$ while $\mathbb{B} = 0$,

on the shell U . Similarly, positive and negative transversal sections satisfy $\mathbb{E}_r = \mathbb{B}_r = 0$ so that the fields $\mathbb{E}$ and $\mathbb{B}$ are tangent to every sphere on the shell. Moreover the vectors $\mathbb{E}$ and $\mathbb{B}$ are orthogonal and of the same length. Their length depends on the radius r only which gives us a function $r \mapsto l(r)$. In the positive transversal case $R_{u,-\pi/2}\mathbb{E}(u) = \mathbb{B}(u)$ for every vector u of the shell, while $R_{u,\pi/2}\mathbb{E}(u) = \mathbb{B}(u)$ in the negative transversal one. The entire class of gauge-invariant sections over U of a given transversal type, is completely determined by this positive function $l = l(r)$.

3.3.2 Radial Solutions

Here we study radial spherically symmetrical solutions of Maxwell equations. Accordingly, let us assume that the scalar fields depend on the radial coordinate only, that is $\epsilon = \epsilon(r)$, $\beta = \beta(r)$ and $\varphi = \varphi(r)$ while similarly for the vectors $\Phi = \mathbb{E} = \mathbb{E}_r(r)e_r$ and $\mathbb{A} = \mathbb{A}_r(r)e_r$. Equations (3.6) further simplify into

$$\begin{aligned} \frac{1}{r^2}(r^2\mathbb{E}_r)' &= \beta\varphi & \epsilon\varphi + \mathbb{A}_r\mathbb{E}_r &= 0 \\ \epsilon' &= \beta\mathbb{A}_r & \beta' + \epsilon\mathbb{A}_r + \varphi\mathbb{E}_r &= 0. \end{aligned} \quad (3.7)$$

We assume that $\mathbb{E}_r$ is given. So we have a system of 4 equations for 4 functions ϵ , β , φ and $\mathbb{A}_r$. The first pair of equations completely fixes the connection components φ and $\mathbb{A}_r$ in terms of ϵ and β . Indeed,

$$\varphi = \frac{1}{\beta r^2}(r^2\mathbb{E}_r)' \quad \mathbb{A}_r = -\frac{\epsilon}{\beta r^2\mathbb{E}_r}(r^2\mathbb{E}_r)'$$

and the second pair of equations now reads

$$\epsilon' = -\frac{\epsilon}{r^2\mathbb{E}_r}(r^2\mathbb{E}_r)' \quad \beta' = \frac{1}{\beta r^2} \left\{ \frac{\epsilon^2}{\mathbb{E}_r} - \mathbb{E}_r \right\} (r^2\mathbb{E}_r)'.$$

This is easily solvable by a direct integration, and we obtain

$$\epsilon = \frac{\lambda}{r^2\mathbb{E}_r} \quad \beta = \pm \sqrt{\Lambda - \frac{\lambda^2}{r^4\mathbb{E}_r^2} - \mathbb{E}_r^2 + 4 \int_r^\infty \frac{\mathbb{E}_u^2}{u} du} \quad (3.8)$$

where λ is an arbitrary real constant and Λ is such, so that the expression under the square root is positive, everywhere.

Remark 3.8. We have assumed here some reasonable behaviour of the electric field $\mathbb{E}$: For example, that it is everywhere non-zero and that the integral in the above expression for β is finite. In fact, the spherical symmetry of $\mathbb{E}$ implies that at the center $r = 0$ the field vanishes (however we should always keep in mind the limitations of the point-particle model). It is also worth mentioning that we can always put $\lambda = 0$ which makes ϵ (and hence $\mathbb{A}_r$) identically zero. Actually, we can always perform a gauge transformation and make $\mathbb{A}_r$ vanish.

Remark 3.9. So the simple solution with $\lambda = 0$ is

$$\beta = \pm \sqrt{\Lambda - \mathbb{E}_r^2 + 4 \int_r^\infty \frac{\mathbb{E}_u^2}{u} du}. \quad (3.9)$$

It is reasonable to assume that furthermore $\Lambda = 0$. This is equivalent to saying that the scalar field vanishes at infinity, assuming that the electric field vanishes at the infinity, in such a way that the integral under the above square root vanishes asymptotically, and also

$$4 \int_r^\infty \frac{\mathbb{E}_u^2}{u} du \geq \mathbb{E}_r^2 \quad (3.10)$$

for all values of r . This inequality becomes (functional, for all r) equality precisely when $\mathbb{E}$ obeys the inverse square law, in which case the scalar field vanishes.

Remark 3.10. The above inequality will be satisfied if $u^2|\mathbb{E}_u| \geq r^2|\mathbb{E}_r|$ whenever $u \geq r$. Indeed,

$$4 \int_r^\infty \frac{\mathbb{E}_u^2}{u} du \geq 4r^4\mathbb{E}_r^2 \int_r^\infty \frac{du}{u^5} = 4r^4\mathbb{E}_r^2 \frac{1}{4} \frac{1}{r^4} = \mathbb{E}_r^2$$

and this can be physically interpreted as a manifestation of a non-point-like charge.

Proposition 3.5. *The total energy of a spherically symmetrical particle-type solution is divided into electro-magnetic and scalar parts in proportion 3 to one.*

Proof. From equation (3.9) we obtain for the total energy

$$\begin{aligned} \frac{1}{2} \iiint (\epsilon^2 + \mathbb{E}^2) dx dy dz &= 4\pi \int_0^\infty r^2 \left\{ 4 \int_r^\infty \frac{\mathbb{E}_u^2 du}{u} \right\} \frac{dr}{2} \\ &= 8\pi \int \int_{r \leq u} r^2 \frac{\mathbb{E}_u^2}{u} du dr = 8\pi \int_0^\infty \frac{u^3}{3} \frac{\mathbb{E}_u^2}{u} du = \frac{4}{3} \frac{1}{2} \iiint \mathbb{E}^2 dx dy dz \end{aligned}$$

and so the proportion between the electromagnetic and scalar total energy is always 3 to one. $\square$

We can arrive to the same solutions by using the matrix form of the Maxwell equations. For static and radially spherically symmetrical configurations we have

$$\mathfrak{D}(\mu, \Phi) = (\text{div}(\Phi), \text{grad}(\mu))$$

and hence the connection matrix is

$$\begin{aligned} (\varphi, \mathbb{A}) &= \frac{i}{\mathbb{D}} (\text{div}(\Phi), \text{grad}(\mu)) (\mu, -\Phi) = \\ &= \frac{i}{\mathbb{D}} (\mu \text{div}(\Phi) - \text{grad}(\mu) \cdot \Phi, \mu \text{grad}(\mu) - \text{div}(\Phi) \Phi). \end{aligned}$$

Because of the symmetricity, we can always represent a connection in such a way that $\mathbb{A} = 0$. The above expression then implies

$$\mu \text{grad}(\mu) = \text{div}(\Phi) \Phi \quad \varphi = i \frac{\text{div} \Phi}{\mu}. \quad (3.11)$$

In general, however, Φ is not necessarily real. So we have

$$\mu \mu' - \frac{1}{r^2} (r^2 \Phi)' \Phi = 0 \quad \Leftrightarrow \quad \mathbb{D}' = \frac{4}{r^2} \Phi^2$$

where we have reinterpreted Φ as its radial component. This immediately gives

$$\mu = \sqrt{\Lambda + \Phi^2 + 4 \int \frac{\Phi^2}{u} du}$$

which is equivalent to the above derived expressions for quantities ϵ and β . It is worth mentioning that in this representation, there is still a non-trivial condition regarding the reality of the scalar potential φ .

Lemma 3.6. *In fact, essentially the same reasoning applies to a class of more general situations, involving static configurations with vanishing connection 3-vector. However there is a non-trivial condition on the charge density ρ . Its gradient must be parallel to Φ .*

Proof. The vanishing of $\mathbb{A}$ implies that the product $\rho\Phi$ is a gradient. Indeed, according to the first equation in (3.11) we have

$$\rho\Phi = \frac{1}{2}\text{grad}(\mu^2)$$

and therefore $\text{rot}(\rho\Phi) = \text{grad}(\rho) \times \Phi = 0$ and the two vectors are parallel. If conversely, the gradient of ρ is parallel to Φ then we can find a complex scalar field μ such that the above equation holds. The solution will be consistent, iff the scalar potential given by the second equation in (3.11) is *real*. Electrostatical configurations with $\mathbb{B} = 0$ are all included, because it is always possible to adjust μ^2 to be negative, so that μ will be imaginary. $\square$

Remark 3.11. To elaborate more on the purely electrostatical case $\Phi = \mathbb{E}$ and $\mathbb{B} = 0$: Far away from the particles, the charge density and electric field intensity (almost) vanish, so μ^2 will be (almost) constant outside of a compact region, and therefore bounded. In particular, we can assume that it is strictly negative, everywhere. It is also worth observing that the above simple consideration also give us an indication which statical charge distributions *are not possible*. Therefore our theory imposes a non-trivial conditions on the charge distributions. An intuitive physical interpretation of this interesting phenomenon, is that particles somehow feel the presence of the external field, and adjust internally in the appropriate way, much like conductors adjusting their charges over the surface in a response to an external electromagnetic change in their environment. So that in the interior of particles (where $\rho \neq 0$) our parallelism condition holds, in all physically relevant situations.

Remark 3.12. Taking into account the identity $\text{rot}^2(\Phi) = \text{grad}(\text{div}(\Phi)) - \Delta(\Phi)$ we see that in the case of potential fields $\text{grad}(\rho) = \Delta(\Phi)$. So we can formulate equivalently the parallelism condition as the property that $\Delta(\Phi)$ and Φ be everywhere collinear. If ϕ is a scalar potential for Φ so that

$\Phi = \text{grad}(\phi)$, yet another operational form of the condition is to say that $\rho = \Delta(\psi) = F(\psi)$ where F are (real and local) smooth functions.

As a concrete illustration, let us imagine a particle-like object whose electric field is of the form

$$\mathbb{E} = \mathbb{E}_r e_r = \frac{f(r)}{r^2} \quad (3.12)$$

where f is a ‘modulation function’, slowly increasing and positive, or decreasing and negative. It provides a correction factor to the model of completely localized point-like particle. Assuming that $\epsilon = 0$ (and hence $\mathbb{A} = 0$), the formulae for the scalar connection component and the scalar field give

$$\beta = \frac{2}{r^2} f(r) \sqrt{\int_1^\infty \left\{ \frac{f(ur)^2}{f(r)^2} - 1 \right\} \frac{du}{u^5}} \quad (3.13)$$

$$\varphi = \frac{1}{2} \frac{f'(r)}{f(r)} \left(\int_1^\infty \left\{ \frac{f(ru)^2}{f(r)^2} - 1 \right\} \frac{du}{u^5} \right)^{-\frac{1}{2}}. \quad (3.14)$$

Remark 3.13. And, in a very particular case of $f(r) = qr^a/4\pi$, where $a > 0$ is a small number, we obtain

$$\int_1^\infty \left\{ \frac{f(ru)^2}{f(r)^2} - 1 \right\} \frac{du}{u^5} = \frac{a}{2-a} \quad \varphi = \frac{1}{2} \sqrt{a(2-a)} \frac{1}{r}.$$

If $a = 0$, in other words if the electric field is classical obeying the inverse square law, we see that μ vanishes and the connection is flat (φ vanishes, too).

Let us now consider a kind of a complementary problem, when a static spherically symmetrical connection ω is given, and we are looking for the appropriate fields μ and Φ . As already indicated a couple of paragraphs above, we can always perform a spatial gauge transformation, and make $\mathbb{A}$ identically zero (as it is a radial field, thus a gradient). Therefore the only possibly non-trivial component of the connection is its scalar potential φ . So in this case Maxwell equations reduce to

$$\text{div}(\Phi) = -i\varphi\mu \quad \text{rot}(\Phi) = i\text{grad}(\mu) - \varphi\Phi. \quad (3.15)$$

The above form of the equations is, in general, equivalent to restricting on those stationary solutions for which $\mathbb{K}$ vanishes identically.

At first, we shall analyze a special case of *potential* (including as its own special case the spherically symmetrical solutions) fields Φ . In this case the rotor vanishes and we find

$$\Delta\left(\frac{\mu}{\sqrt{\varphi}}\right) = \left\{ \sqrt{\varphi}\Delta\left(\frac{1}{\sqrt{\varphi}}\right) - \varphi^2 \right\} \frac{\mu}{\sqrt{\varphi}} \quad \Phi = \frac{i}{\varphi} \text{grad}(\mu) \quad (3.16)$$

where we have assumed that φ is non-zero everywhere (so it is either positive or negative, and let us assume to simplify things, that it is positive). Thus we have to solve one second-order differential equation for the primary field μ , a kind of a generalized spectral equation for the Laplacian, and Φ is then given by the second formula above. If now φ is spherically symmetrical, then we can decouple the radial and spherical variables and write

$$\frac{\mu}{\sqrt{\varphi}} = \frac{S(r)}{r} Y_{lm}(\theta, \phi)$$

where Y_{lm} are the spherical harmonics. The differential equation for μ becomes

$$S'' + \left\{ \varphi^2 - \sqrt{\varphi}\Delta\left(\frac{1}{\sqrt{\varphi}}\right) - \frac{l(l+1)}{r^2} \right\} S = 0 \quad (3.17)$$

as it follows from the polar coordinates form of the Laplacian.

An important special case of such configurations arises when we assume that the entire expression

$$\kappa^2 = \varphi^2 - \sqrt{\varphi}\Delta\left(\frac{1}{\sqrt{\varphi}}\right)$$

is constant. Then the equation for μ is the eigenproblem for the Laplacian, and every solution gives a consistent Φ . This will be the case, in particular, when φ is constant. The solutions for $l = 0$ are

$$\mu \sim \frac{\sin(\kappa r)}{r} \quad \mu \sim \frac{\cos(\kappa r)}{r} \quad (3.18)$$

and for general l we obtain the solutions in a form of hypergeometric expansions

$$\begin{aligned} S &= \sum_{m \geq 0} \frac{(-)^m \kappa^{2m} r^{l+1+2m}}{2^m m! (2l + 2m + 1)!!} \\ S &= \sum_{m \geq 0} \frac{(-)^m \kappa^{2m} r^{2m-l}}{(2m)!! (2m - 2l - 1)!!} \end{aligned} \quad (3.19)$$

of complementary parity. By the way, it is interesting to observe the symmetry between these solutions, and how in the case $l = 0$ the above formulae reduce to the standard power series expansions for sin and cos.

Let us now focus on decoupling equations (3.15) in general.

Lemma 3.7. *These general (for the context given) equations imply*

$$\Delta(\Phi) = \text{grad}(\varphi) \times \Phi - \varphi^2 \Phi - i\mu \text{grad}(\varphi). \quad (3.20)$$

Proof. Substituting μ from the first equation in (3.15) into the second equation, we obtain

$$\begin{aligned} \text{rot}(\Phi) + \varphi \Phi &= -\text{grad} \left[\frac{\text{div}(\Phi)}{\varphi} \right] = \frac{1}{\varphi^2} \text{div}(\Phi) \text{grad}(\varphi) - \frac{1}{\varphi} \text{grad}[\text{div}(\Phi)] \\ &= -\frac{1}{\varphi} \Delta(\Phi) - \frac{1}{\varphi} \text{rot}^2(\Phi) + \frac{1}{\varphi^2} \text{div}(\Phi) \text{grad}(\varphi) \\ &= \frac{1}{\varphi} \text{rot}(\varphi \Phi) - \frac{1}{\varphi} \Delta(\Phi) + \frac{1}{\varphi^2} \text{div}(\Phi) \text{grad}(\varphi). \end{aligned}$$

Further elementary transformations with identities

$$\text{rot}(\varphi \Phi) = \varphi \text{rot}(\Phi) + \text{grad}(\varphi) \times \Phi \quad \text{rot}^2(\Phi) = \text{grad}[\text{div}(\Phi)] - \Delta(\Phi)$$

as well as a consequence $\text{rot}^2(\Phi) = -\text{rot}(\varphi \Phi)$ of the second equation in (3.15), lead directly to the result. $\square$

3.3.3 Transversal Solutions

Our second class of spherically-symmetrical solutions is very different. It is given by configurations in which electric and magnetic fields are transversal,

in other words $\mathbb{E}$ and $\mathbb{B}$ are perpendicular to e_r . In order to achieve the spherical symmetry of the complete configuration $\Phi = \mathbb{E} + i\mathbb{B}$, both fields $\mathbb{E}$ and $\mathbb{B}$ should have *the same intensity*, and be *perpendicular* to each other. Because of the gauge invariance of our theory, we are free to choose these fields in a way particularly comfortable for performing calculations. So to begin with, in a chosen polar coordinate system, let us assume that

$$\mathbb{E}_r = \mathbb{E}_\phi = 0 \quad \mathbb{B}_r = \mathbb{B}_\theta = 0 \quad \mathbb{E}_\theta = \mathbb{B}_\phi = h(r). \quad (3.21)$$

By expanding the divergence and rotor in spherical coordinates, we obtain

$$\begin{aligned} \operatorname{div}(\mathbb{E}) &= \frac{h(r)}{r} \cot(\theta) & \operatorname{div}(\mathbb{B}) &= 0 \\ \operatorname{rot}(\mathbb{B}) &= \frac{\cot(\theta)}{r} h(r) e_r - \frac{1}{r} \frac{\partial}{\partial r} [r h(r)] e_\theta & \operatorname{rot}(\mathbb{E}) &= \frac{1}{r} \frac{\partial}{\partial r} [r h(r)] e_\phi \end{aligned}$$

and hence

$$\frac{\partial}{\partial r} [r h(r)] = 0 \quad \mathbb{A}_\phi = \mathbb{A}_r = 0 \quad \mathbb{A}_\theta = \frac{1}{r} \cot(\theta)$$

which implies $h(r) = \kappa/r$ and the following form for electric and magnetic fields

$$\mathbb{E} = \frac{\kappa}{r} e_\theta \quad \mathbb{B} = \frac{\kappa}{r} e_\phi \quad (3.22)$$

where κ is a constant parameter. The curvature field is given by

$$g = 0 \quad \mathbb{K} = \operatorname{rot}(\mathbb{A}) = -\frac{1}{r^2} e_r. \quad (3.23)$$

The energy density and the Poynting vector are given by

$$w = \frac{\kappa^2}{r^2} \quad \mathbb{P} = c \frac{\kappa^2}{r^2} e_r \quad (3.24)$$

as is easily seen by looking at the expression for the electric and magnetic field.

There is also a dual spherically symmetrical solution, represented by

$$\mathbb{E}_r = \mathbb{E}_\phi = 0 \quad \mathbb{B}_r = \mathbb{B}_\theta = 0 \quad -\mathbb{E}_\theta = \mathbb{B}_\phi = h(r). \quad (3.25)$$

In this case $\mathbb{A}$ and $\mathbb{K} = \text{rot}(\mathbb{A})$, as well as $\mathbb{P}$ change the sign

$$\mathbb{P} = -c \frac{\kappa^2}{r^2} e_r \quad \mathbb{A} = -\frac{1}{r} \cot(\theta) e_\phi \quad \mathbb{K} = \frac{1}{r^2} e_r \quad (3.26)$$

while the energy density w and the graviational field g remain unchanged.

Remark 3.14. The first of these transversal spherically symmetric solutions looks like an ‘exploding microstar’: It appears that the origin O is a source of a uniform and radial flow of electromagnetic field. The second solution is just the opposite ‘imploding’ version.

4 Concluding Remarks

The theory we have presented, builds on the idea of an intrinsic and completely localizable symmetry between electric and magnetic fields. It turns out that such a symmetry can be introduced consistently, only if we refine the very concept of symmetry—as a transformation not only acting on the underlying structure and then propagating to higher geometrical orders, but as a possibly autonomous change occurring at any level. For us, the principal illustration is the reflection symmetry which changes the complex structure on the electromagnetic field amplitudes space.

Among diverse natural conditions we have found for such an extended electromagnetic field, we can highlight here the regularity condition, requiring the invertibility of the field matrix. Without any lack of observational generality, we can assume that all configurations are such. Indeed, the invertible fields form an everywhere dense open set in the space of all electromagnetic configurations. On the other hand, the invertible configurations are interpretable as endomorphisms of the associated projective bundle, to the underlying 2-dimensional complex vector bundle providing the habitat for the initial electromagnetic field configurations. An additional benefit from this interpretation, is that projective endomorphisms are associated to classes of gauge equivalent electromagnetic configurations. Every such a class contains an additional information, given by the absolute value of the determinant $\mathbb{D} = \mu^2 - \Phi \cdot \Phi$.

Our theory requires an additional refinement—a system of differential equations governing the connection field. So far we have only considered a naturally extended and generalized Maxwell equations, without discussing any dynamics of the connection field. This will be one of the principal topics of the second part of this study. As we shall see, there exists a natural solution, which postulates a similar to the Maxwell equations first-order behaviour of the connection field: The curvature tensor, which is an invariant combination of the connection field partial derivatives, should be expressible as a quadratic covariant combination of the vector and scalar electromagnetic fields. This leads to a system of first order equations, which implies, via the boundary conditions at infinity, the quantization of electromagnetic charge (much in conceptual resonance with the Schrödinger solution of the Hydrogen Atom).

In such a way, the equations will resume an incredible property of the standard Maxwell equations in vacuum: the time derivative of the fields is encoded in the spatial configuration. This, in turn, allows us develop a new interpretation of time, fully in resonance with Mach contextual mechanical philosophy. In particular, a timeless Platonia playground of the kind of [1], emerges naturally.

Another elegant conceptual and timeless framework, from which the space-time structure emerges, is developed in [11], where the basic structure is given by the category of physical objects and their relative velocities understood as primary structure, given by morphisms of this category which is a groupoid category (every morphism is an isomorphism). It would be very interesting to see how the presented electromagnetism fits into this categorical framework, and what would be the counterparts of our basic gauge-invariant equations. In particular, as discussed in detail in [12], the groupoid covariance leads to a relativity theory, with electric and magnetic fields, different from the standard formulation based on the Lorentz and Poincaré group covariance. Thus, it looks natural to expect that the symmetric electromagnetism would also manifest different, and possibly new, properties within a groupoid-covariant relativity.

Our symmetric electromagnetism, promotes a Platonic interpretation of forces—as things secondary to the primary field, the electromagnetic field. In particular, the gravitational force can be viewed as a derived phenomena,

occurring within the context of mutual geometrical interactions of different dynamical oscillations.

The initial global symmetry $SO(2)$ between electric and magnetic fields can be localized via a standard Yang-Mills mechanism. The corresponding connection form, interpreted as a dynamical field, is a kind of a symbiosis of the Heaviside gravitational field and a form-generating force. Observable physical quantities are in one-to-one correspondence with gauge-invariant geometric objects. In our theory the particles are secondary objects—stable spherically symmetrical configurations involving electromagnetic and connection fields. Both fields are necessary, in order to construct such a stable configuration.

It turns out that the equations of motions for the localized field particles-type objects, are in fact, the standard Einstein-Newton equations of motion. These equations are not postulated, but derived as a consequence of the pure symmetric electromagnetism equations.

And all of this was envisioned by Nikola Tesla [15]: gravitational forces and all forces and visible matter, are of the pure electromagnetic origin. In forthcoming papers we shall develop a dynamics for such particles, together with a quantum geometric generalization of the theory, formulated within the conceptual framework of quantum principal bundles [3, 4, 5].

A Principal Vector Identities

Let us describe the divergence, rotor and Laplacian in cylindric and polar coordinates. Let us consider a generic vector field $\mathbb{X}$. In cylindric coordinates (ρ, ϕ, z) we can write

$$\begin{aligned} \mathbb{X} &= \mathbb{X}_\rho e_\rho + \mathbb{X}_\phi e_\phi + \mathbb{X}_z e_z \\ e_\rho &= \frac{\partial}{\partial \rho} \quad e_\phi = \frac{1}{\rho} \frac{\partial}{\partial \phi} \quad e_z = \frac{\partial}{\partial z} \end{aligned} \tag{A.1}$$

and the operators of divergence and rotor assume the form

$$\begin{aligned} \operatorname{div}(\mathbb{X}) &= \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho \mathbb{X}_\rho) + \frac{1}{\rho} \frac{\partial \mathbb{X}_\phi}{\partial \phi} + \frac{\partial \mathbb{X}_z}{\partial z} \\ \operatorname{rot}(\mathbb{X}) &= \left\{ \frac{1}{\rho} \frac{\partial \mathbb{X}_z}{\partial \phi} - \frac{\partial \mathbb{X}_\phi}{\partial z} \right\} e_\rho + \left\{ \frac{\partial \mathbb{X}_\rho}{\partial z} - \frac{\partial \mathbb{X}_z}{\partial \rho} \right\} e_\phi + \left\{ \frac{\partial}{\partial \rho} (\rho \mathbb{X}_\phi) - \frac{\partial \mathbb{X}_\rho}{\partial \phi} \right\} \frac{e_z}{\rho} \end{aligned}$$

while the Laplacian is given by

$$\Delta = \nabla^2 = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left[\rho \frac{\partial}{\partial \rho} \right] + \frac{1}{\rho^2} \frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}. \quad (\text{A.2})$$

On the other hand, in polar coordinates (r, θ, ϕ) we have

$$\begin{aligned} \mathbb{X} &= \mathbb{X}_r e_r + \mathbb{X}_\theta e_\theta + X_\phi e_\phi \\ e_r &= \frac{\partial}{\partial r} \quad e_\theta = \frac{1}{r} \frac{\partial}{\partial \theta} \quad e_\phi = \frac{1}{r \sin(\theta)} \frac{\partial}{\partial \phi} \end{aligned} \quad (\text{A.3})$$

and the divergence is given by

$$\operatorname{div}(\mathbb{X}) = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \mathbb{X}_r) + \frac{1}{r \sin(\theta)} \frac{\partial}{\partial \theta} (\sin(\theta) \mathbb{X}_\theta) + \frac{1}{r \sin(\theta)} \frac{\partial X_\phi}{\partial \phi} \quad (\text{A.4})$$

while the rotor operator assumes the form

$$\begin{aligned} \frac{1}{r \sin(\theta)} \left\{ \frac{\partial}{\partial \theta} (\sin(\theta) X_\phi) - \frac{\partial \mathbb{X}_\theta}{\partial \phi} \right\} e_r + \frac{1}{r} \left\{ \frac{1}{\sin(\theta)} \frac{\partial \mathbb{X}_r}{\partial \phi} - \frac{\partial}{\partial r} (r \mathbb{X}_\phi) \right\} e_\theta \\ + \frac{1}{r} \left\{ \frac{\partial}{\partial r} (r \mathbb{X}_\theta) - \frac{\partial \mathbb{X}_r}{\partial \theta} \right\} e_\phi = \operatorname{rot}(\mathbb{X}). \end{aligned} \quad (\text{A.5})$$

Finally, here is the Laplacian in spherical coordinates:

$$\Delta = \nabla^2 = \frac{1}{r^2} \frac{\partial}{\partial r} \left[r^2 \frac{\partial}{\partial r} \right] + \frac{1}{r^2 \sin(\theta)} \frac{\partial}{\partial \theta} \left[\sin(\theta) \frac{\partial}{\partial \theta} \right] + \frac{1}{r^2 \sin(\theta)^2} \frac{\partial^2}{\partial \phi^2}.$$

Let us now assume that a pair of fields $\mathbb{X}$ and $\mathbb{Y}$ is given, transforming according to the adjoint action of $M(2)$. From the inertial system in which

this configuration is moving with the uniform velocity v the transformed fields will be

$$\begin{aligned}\mathbb{X}_{||} &\rightsquigarrow \mathbb{X}_{||} & \mathbb{X}_{\perp} &\rightsquigarrow \frac{\mathbb{X}_{\perp} - v \times \mathbb{Y}_{\perp}/c}{\sqrt{1 - v^2/c^2}} \\ \mathbb{Y}_{||} &\rightsquigarrow \mathbb{Y}_{||} & \mathbb{Y}_{\perp} &\rightsquigarrow \frac{\mathbb{Y}_{\perp} + v \times \mathbb{X}_{\perp}/c}{\sqrt{1 - v^2/c^2}}.\end{aligned}\tag{A.6}$$

The basic invariants are given by the real and imaginary part of the scalar product of the complex vector $\mathbb{X} - i\mathbb{Y}$ with itself

$$(\mathbb{X} - i\mathbb{Y})^2 = \mathbb{X}^2 - \mathbb{Y}^2 - 2i\mathbb{X} \cdot \mathbb{Y}.\tag{A.7}$$

If in a given space-time point $\mathbb{X}$ and $\mathbb{Y}$ are both non-zero and parallel, then the only boosts such that the transformed vectors would remain parallel are those with v parallel to the direction defined by $\mathbb{X}$ and $\mathbb{Y}$. Indeed, if $\mathbb{X} = \kappa\mathbb{Y}$ then

$$\frac{\mathbb{X}_{\perp} - v \times \mathbb{Y}_{\perp}/c}{\sqrt{1 - v^2/c^2}} = \kappa \frac{\mathbb{Y}_{\perp} + v \times \mathbb{X}_{\perp}/c}{\sqrt{1 - v^2/c^2}}$$

which is only possible for $\mathbb{X}_{\perp} = \mathbb{Y}_{\perp} = 0$ in other words the relative velocity vector v should be parallel to $\mathbb{X}$ and $\mathbb{Y}$. On the other hand, for every non-parallel $\mathbb{X}$ and $\mathbb{Y}$ having a non-zero complex length square (A.7) there exists a vector v such that the corresponding boost makes $\mathbb{X}$ and $\mathbb{Y}$ parallel. Such a solution can always be found so that v is perpendicular to $\mathbb{X}$ and $\mathbb{Y}$. From the above equation we find

$$\mathbb{X} - \kappa\mathbb{Y} = v \times (\kappa\mathbb{X} + \mathbb{Y})/c$$

so κ is obtained from the orthogonality condition of $\mathbb{X} - \kappa\mathbb{Y}$ and $\kappa\mathbb{X} + \mathbb{Y}$. This gives

$$\kappa = \frac{\mathbb{X}^2 - \mathbb{Y}^2 \pm \sqrt{\mathbb{X}^2 + \mathbb{Y}^2 - 4(\mathbb{X} \times \mathbb{Y})^2}}{2\mathbb{X} \cdot \mathbb{Y}}$$

for non-orthogonal $\mathbb{X}$ and $\mathbb{Y}$, in which case κ we should choose κ to be the positive or negative solution, depending if the angle between $\mathbb{X}$ and $\mathbb{Y}$ is sharp or obtuse (as we should always have $|\mathbb{X} - \kappa\mathbb{Y}| < |\kappa\mathbb{X} + \mathbb{Y}|$). If $\mathbb{X}$ and $\mathbb{Y}$ are orthogonal, the solution is simply $\kappa = 0$.

Now we shall prove that a necessary and sufficient condition for the object

$$p(\mathbb{X}, \mathbb{Y}) = ((\mathbb{X}^2 + \mathbb{Y}^2)/2, c\mathbb{X} \times \mathbb{Y})$$

to constitute a 4-vector is that the quadratic invariant (A.7) vanishes. In other words the vectors $\mathbb{X}$ and $\mathbb{Y}$ are *orthogonal* and of the *same length*. Let us observe that in general

$$p(\mathbb{X}, \mathbb{Y})^2 = \frac{c^2}{4} |(\mathbb{X} - i\mathbb{Y})^2|^2. \quad (\text{A.8})$$

If the complex square of $\mathbb{X} - i\mathbb{Y}$ is non-zero, there exists a one-parameter family of boosts, such that the transformed $\mathbb{X}$ and $\mathbb{Y}$ are parallel in other words $\mathbb{X} \times \mathbb{Y}$ vanishes. This is incompatible with a 4-vector transformational behavior. On the other hand, if the invariant is zero, then

$$w(\mathbb{X}, \mathbb{Y}) = \frac{1}{2}(\mathbb{X}^2 + \mathbb{Y}^2).$$

We would like to calculate the quantity

$$W = \frac{1}{2} \iiint (\mathbb{X}^2 + \mathbb{Y}^2) dx dy dz = W_0 + \iiint \mathbb{Y}^2 dx dy dz. \quad (\text{A.9})$$

Let us now assume that $\mathbb{X} = \xi(r)e_r$ is spherically symmetrical and $\mathbb{Y} = 0$. The transformed $\mathbb{X}$ will still look radial, although not spherically symmetrical as it will contain an explicit angular dependency on θ , and an additional magnetic component $\mathbb{Y}$ will appear, so that

$$\mathbb{Y} = \frac{v}{c} \times \mathbb{X} \quad \mathbb{X} = \frac{\xi \left\{ r \sqrt{\frac{1 - \sin^2(\theta)v^2/c^2}{1 - v^2/c^2}} \right\}}{\sqrt{1 - \sin^2(\theta)v^2/c^2}} e_r. \quad (\text{A.10})$$

Let us calculate explicitly the integrals of the transformed squares $\mathbb{X}^2$ and

$\mathbb{Y}^2$ over the physical 3-space. We have

$$\begin{aligned}
 \frac{1}{2} \iiint \mathbb{X}^2 dx dy dz &= \iiint \frac{\xi \left\{ r \sqrt{\frac{1 - \sin^2(\theta) v^2 / c^2}{1 - v^2 / c^2}} \right\}^2}{1 - \sin^2(\theta) v^2 / c^2} dx dy dz = \\
 &= \pi \int_0^\pi \int_0^\infty \sin(\theta) \frac{\xi \left\{ r \sqrt{\frac{1 - \sin^2(\theta) v^2 / c^2}{1 - v^2 / c^2}} \right\}^2}{1 - \sin^2(\theta) v^2 / c^2} r^2 dr d\theta = \\
 &= \frac{1}{2} W_0 \int_0^\pi \left(\frac{1 - v^2 / c^2}{1 - \sin^2(\theta) v^2 / c^2} \right)^{3/2} \frac{\sin(\theta) d\theta}{1 - \sin^2(\theta) v^2 / c^2} = \\
 &= \frac{1}{2} W_0 (1 - v^2 / c^2)^{3/2} \int_{-1}^1 \frac{dt}{(t^2 v^2 / c^2 + 1 - v^2 / c^2)^{5/2}} = \left(1 - \frac{1}{3} \frac{v^2}{c^2} \right) \frac{W_0}{\sqrt{1 - \frac{v^2}{c^2}}}
 \end{aligned}$$

and similarly for the second integral

$$\begin{aligned}
 \frac{1}{2} \iiint \mathbb{Y}^2 dx dy dz &= \frac{v^2}{c^2} \iiint \frac{\xi \left\{ r \sqrt{\frac{1 - \sin^2(\theta) v^2 / c^2}{1 - v^2 / c^2}} \right\}^2}{1 - \sin^2(\theta) v^2 / c^2} \sin^2(\theta) dx dy dz \\
 &= \pi \frac{v^2}{c^2} \int_0^\pi \int_0^\infty \sin^3(\theta) \frac{\xi \left\{ r \sqrt{\frac{1 - \sin^2(\theta) v^2 / c^2}{1 - v^2 / c^2}} \right\}^2}{1 - \sin^2(\theta) v^2 / c^2} r^2 dr d\theta \\
 &= \frac{1}{2} W_0 \frac{v^2}{c^2} \int_0^\pi \left(\frac{1 - v^2 / c^2}{1 - \sin^2(\theta) v^2 / c^2} \right)^{3/2} \frac{\sin^3(\theta) d\theta}{1 - \sin^2(\theta) v^2 / c^2} \\
 &= \frac{1}{2} W_0 (1 - v^2 / c^2)^{3/2} \frac{v^2}{c^2} \int_{-1}^1 \frac{(1 - t^2) dt}{(t^2 v^2 / c^2 + 1 - v^2 / c^2)^{5/2}} = \frac{2}{3} \frac{v^2}{c^2} \frac{W_0}{\sqrt{1 - \frac{v^2}{c^2}}}.
 \end{aligned}$$

Finally, the calculation for the integral of the vector product expression

$c\mathbb{X} \times \mathbb{Y} = \mathbb{X}^2 v - (v \cdot \mathbb{X})\mathbb{X}$ gives

$$\begin{aligned} \iiint \mathbb{X} \times \mathbb{Y} \, dx dy dz &= \frac{v}{c} \iiint \frac{\xi \left\{ r \sqrt{\frac{1 - \sin^2(\theta) v^2/c^2}{1 - v^2/c^2}} \right\}^2}{1 - \sin^2(\theta) v^2/c^2} (1 - \cos^2(\theta)) \\ dx dy dz &= W_0 \sqrt{1 - \frac{v^2}{c^2}}^3 \frac{v}{c} \int_{-1}^1 \frac{(1 - t^2) dt}{(t^2 v^2/c^2 + 1 - v^2/c^2)^{5/2}} = \frac{4}{3} \frac{W_0}{\sqrt{1 - \frac{v^2}{c^2}}} \frac{v}{c}. \end{aligned}$$

In the above calculations we have used the standard integrals

$$\int_{-1}^1 \frac{dt}{(\lambda^2 - 1 + t^2)^{5/2}} = \frac{2}{3} \frac{(3\lambda^2 - 1)}{\lambda^3(\lambda^2 - 1)^2} \quad \int_{-1}^1 \frac{(1 - t^2) dt}{(\lambda^2 - 1 + t^2)^{5/2}} = \frac{4}{3} \frac{1}{\lambda(\lambda^2 - 1)^2}$$

with $\lambda = c/v > 1$.

B On Symmetries

B.1 Global Gauge Transformations

Let us consider a Lie group G and let P be a principal G -bundle over its base manifold M . The gauge transformations of P are defined as *vertical automorphisms* of the bundle P . In other words, they are the smooth maps $\psi: P \rightarrow P$ satisfying

$$\psi(pg) = \psi(p)g \quad \Pi(\psi(p)) = \Pi(p) \quad (\text{B.1})$$

where $\Pi: P \rightarrow M$ is the projection map, while $p \in P$ and $g \in G$. By definition, all gauge transformations form a group $\mathcal{G} = \mathcal{G}(P)$. Since gauge transformations map every vertical fiber of P into itself, they are determined by functions $\gamma: P \rightarrow G$ so that

$$\psi(p) = p\gamma(p) \quad (\text{B.2})$$

for every $p \in P$. The equivariance condition above is equivalent to

$$\gamma(pg) = g^{-1}\gamma(p)g \quad \forall p \in P, \quad \forall g \in G. \quad (\text{B.3})$$

We can therefore write $\psi = \psi_\gamma$ and $\gamma = \gamma_\psi$. In terms of this identification, the composition of gauge transformations is represented as *the pointwise product* of the corresponding maps. In other words

$$\gamma_{\psi\xi} = \gamma_\psi \gamma_\xi \quad \forall \xi, \psi \in \mathcal{G}. \quad (\text{B.4})$$

Indeed, we have

$$\psi\xi(p) = p\gamma_{\psi\xi}(p) = \psi[p\gamma_\xi(p)] = \psi(p)\gamma_\xi(p) = p\gamma_\psi(p)\gamma_\xi(p)$$

and thus the above property holds. It is also worth noticing that the gauge transformations can be naturally interpreted as sections of the adjoint bundle. If F is a smooth manifold and $\rho: G \times F \rightarrow F$ a left action of G on F then the associated bundle $E = E(P, \rho, F)$ is defined as

$$E = P \times_G F \quad (\text{B.5})$$

where the right action of G on $P \times F$ is given by $g: (p, \eta) \mapsto (pg, \rho(g)^{-1}\eta)$. The smooth sections of the bundle E are naturally interpretable as the smooth maps $\zeta: P \rightarrow F$ such that

$$\zeta(pg) = \rho(g)^{-1}\zeta(p)$$

for every $p \in P$ and $g \in G$. Explicitly, the correspondence between such maps and the same objects as sections $\zeta: M \rightarrow E$ is given by

$$[(p, \zeta(p))] = \zeta[\Pi(p)] \quad \forall p \in P.$$

Therefore, in the special case of $F = G$ and G acting on itself by means of the adjoint representation $\rho(g)h = ghg^{-1}$ we obtain exactly the maps γ labeling the gauge transformations. In other words, the group $\mathcal{G}$ can be viewed as the group of the smooth sections of the adjoint bundle $\text{ad}(P) = E(P, \rho, G)$. Here it is worth observing that, since the adjoint action is the action by group automorphisms, each fiber of the adjoint bundle is naturally endowed with a group structure, every point p of the fiber inducing a group isomorphism between this fiber and G .

In the special case of an Abelian G we have the following natural identification:

$$\mathcal{G}(P) \leftrightarrow C^\infty(M, G)$$

and in particular for principal bundles by oriented circles, in other words when $G = \text{SO}(2)$ the situation further simplifies and we have

$$\mathcal{G} \leftrightarrow C^\infty(M, \text{SO}(2)) = \text{U}[C^\infty(M)] \quad (\text{B.6})$$

in other words, gauge transformations are unitary elements of the $*$ -algebra $\mathcal{V} = C^\infty(M)$ of complex smooth functions over M . In such a way, the concept of a gauge transformation is extendible to quantum principal $\text{SO}(2)$ -bundles, they are simply the elements of the unitary group $\text{U}(\mathcal{V})$ of the (now in general) non-commutative $*$ -algebra $\mathcal{V}$ describing the quantum base manifold M .

B.2 Relativistic Invariance

The proper Lorentz group is naturally isomorphic to the Moebius group $\text{M}(2)$ consisting of all automorphisms of the holomorphic 2-sphere. The physical interpretation of this sphere, is that it represents the physical sky built from light rays. In this sense it is a geometrical invariant of the Minkowski space. The universal covering group of $\text{M}(2)$ is $\text{SL}(2, \mathbb{C})$ and we have

$$\text{SL}(2, \mathbb{C})/\{1, -1\} = \text{M}(2).$$

The Lorentz group is also realizable as the group of special orthogonal transformations of the space $\mathbb{C}^3$, equipped with the natural symmetric bilinear form

$$(\xi, \zeta) = \xi_1 \zeta_1 + \xi_2 \zeta_2 + \xi_3 \zeta_3.$$

In other words

$$\text{M}(2) \leftrightarrow \text{SO}(3, \mathbb{C}).$$

The realization is constructed in the following way. In the complex Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ there exists a natural basis

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

given by the Pauli matrices σ_1 , σ_2 and σ_3 . Let us identify vector spaces $\mathfrak{sl} \leftrightarrow \mathbb{C}^3$ using this basis.

The complex symmetric bilinear form

$$(\xi, \zeta) = \frac{1}{8} \text{tr} \{ \text{ad}(\xi) \text{ad}(\zeta) \}$$

is invariant under the adjoint action of $\text{SL}(2, \mathbb{C})$ on its Lie algebra $\mathfrak{sl}(2, \mathbb{C})$. It is just a rescaled Cartan form.

It is easy to check that Pauli matrices are orthonormal with respect to the introduced bilinear form. Since the kernel of the adjoint action of $\text{SL}(2, \mathbb{C})$ consists of matrices 1 and -1 , the adjoint action factorizes to $\text{M}(2)$ and the image of the representation are precisely the elements of $\text{SO}(3, \mathbb{C})$.

Now let us consider the oriented Lorentz frames bundle Q over the Minkowski space-time M . It is a principal $\text{M}(2)$ -bundle over M . The associated vector bundle $\mathcal{E}$, to the above adjoint representation of $\text{M}(2)$ in $\mathbb{C}^3$ provides a natural geometrical framework for studying the configurations Φ of the complex electromagnetic field, which are interpretable as smooth sections of $\mathcal{E}$. Let P be a principal $\text{SO}(2)$ bundle over M . Then we can construct a product bundle $P \times_M Q$, which is a principal $\text{SO}(2) \times \text{M}(2)$ -bundle over M . The product group $\text{SO}(2) \times \text{M}(2)$ is faithfully represented in the space $\mathbb{C}^3$: The rotations from $\text{SO}(2)$ act as unitary complex numbers. In this sense, the action of the circle $\text{SO}(2)$ comes from the complex structure in the space $\mathbb{C}^3$. The product group can be equivalently described as linear transformations of $\mathbb{C}^3$ that preserve the complex symmetric scalar product, modulo a phase factor.

On the other hand, there exists a natural realization of the complex vector Minkowski space $\mathbb{C}^4$ by 2×2 matrices $M_2(\mathbb{C})$. It is explicitly given by

$$(t, x, y, z) \leftrightarrow \begin{pmatrix} t+z & x-iy \\ x+iy & t-z \end{pmatrix} = t + x\sigma_1 + y\sigma_2 + z\sigma_3.$$

In terms of this identification, the complex conjugation corresponds to the matrix adjoint operation, the real part $\mathbb{R}^4$ is given by hermitian matrices, the Minkowski metric corresponds to the determinant and the action of the elements $g \in \text{SL}(2, \mathbb{C})$ is simply

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mapsto g \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} g^\dagger.$$

This factorizes through $\{1, -1\}$ down to $M(2)$ giving the standard Lorentz transformations. On the other hand, the above mentioned adjoint action of $M(2)$ on $\mathbb{C}^3$ can be described in terms of $SL(2, \mathbb{C})$ as

$$\begin{pmatrix} \Phi_3 & \Phi_1 - i\Phi_2 \\ \Phi_1 + i\Phi_2 & -\Phi_3 \end{pmatrix} \mapsto g \begin{pmatrix} \Phi_3 & \Phi_1 - i\Phi_2 \\ \Phi_1 + i\Phi_2 & -\Phi_3 \end{pmatrix} g^{-1}.$$

It also factorizes through $\{1, -1\}$ down to $M(2)$. We see that the matrix product in $M_2(\mathbb{C})$ gives rise to a natural equivariant map between $\mathbb{C}^3 \otimes \mathbb{C}^4$ and $\mathbb{C}^4$. Let us explicitly calculate this pairing in terms of the identifications of $\mathbb{C}^3$ as the electromagnetic field values and $\mathbb{C}^4$ as the complex spacetime coordinates. We have

$$\begin{aligned} \Phi \otimes (t, x, y, z) &\mapsto \begin{pmatrix} \Phi_3 & \Phi_1 - i\Phi_2 \\ \Phi_1 + i\Phi_2 & -\Phi_3 \end{pmatrix} \begin{pmatrix} t + z & x - iy \\ x + iy & t - z \end{pmatrix} = \\ &= (\Phi_1\sigma_1 + \Phi_2\sigma_2 + \Phi_3\sigma_3)(t + x\sigma_1 + y\sigma_2 + z\sigma_3) = \\ &= (x, y, z) \cdot \Phi + \left\{ t\Phi + i\Phi \times (x, y, z) \right\} \cdot \vec{\sigma} \leftrightarrow \left((x, y, z) \cdot \Phi, t\Phi - i(x, y, z) \times \Phi \right). \end{aligned}$$

In a similar way we can introduce dual complex 4-vectors, they generate the space $\mathbb{C}_4$ where $SL(2, \mathbb{C})$ acts in the following way

$$(t, x, y, z) \leftrightarrow \begin{pmatrix} t + z & x - iy \\ x + iy & t - z \end{pmatrix} = X \mapsto (g^\dagger)^{-1} X g^{-1}.$$

Clearly, this is just the dual action of the action introduced on standard (contravariant) complex vectors of the Minkowski space. To put it in slightly different words, there is a natural pairing between the covariant vectors $\mathbb{C}_4$ and electromagnetic field vectors $\mathbb{C}^3$, with values in $\mathbb{C}_4$. In explicit terms:

$$\begin{aligned} (t, x, y, z) \otimes \Phi &\mapsto \begin{pmatrix} t + z & x - iy \\ x + iy & t - z \end{pmatrix} \begin{pmatrix} \Phi_3 & \Phi_1 - i\Phi_2 \\ \Phi_1 + i\Phi_2 & -\Phi_3 \end{pmatrix} = \\ &= (t + x\sigma_1 + y\sigma_2 + z\sigma_3)(\Phi_1\sigma_1 + \Phi_2\sigma_2 + \Phi_3\sigma_3) = \\ &= (x, y, z) \cdot \Phi + \left\{ t\Phi + i(x, y, z) \times \Phi \right\} \cdot \vec{\sigma} \leftrightarrow \left((x, y, z) \cdot \Phi, t\Phi + i(x, y, z) \times \Phi \right). \end{aligned}$$

Thus there exists an intrinsic pairing between the sections of the corresponding vector bundles. We can also use a symbolic substitution (∇_0, ∇)

for covariant vectors, and $(\nabla_0, -\nabla)$ for contravariant ones, and thus conclude that the following expressions

$$\begin{aligned}\Phi &\mapsto \Phi \otimes (\nabla_0, -\nabla) \mapsto (-\nabla \cdot \Phi, \nabla_0 \Phi + i\nabla \times \Phi) \\ \Phi &\mapsto (\nabla_0, \nabla) \otimes \Phi \mapsto (\nabla \cdot \Phi, \nabla_0 \Phi + i\nabla \times \Phi)\end{aligned}\quad (\text{B.7})$$

provide intrinsic contravariant and covariant complex vector fields, respectively, built from an arbitrary electromagnetic field configuration Φ .

The presented scheme allows a natural extension to include scalar fields represented by scalar matrices. Indeed the adjoint action of $\text{SL}(2, \mathbb{C})$ is naturally defined on the whole of $M_2(\mathbb{C}) = \mathbb{C} \oplus \mathfrak{sl}(2, \mathbb{C})$ reducing to the trivial action on scalar matrices.

Let us now compute the matrices representing the action of the field (μ, Φ) on $\mathbb{C}^4$, in the standard coordinates of the complexified Minkowski space. The pair (μ, Φ) is acting on the left, and the pair $(\bar{\mu}, \bar{\Phi}) = (\mu, \Phi)^*$ is acting on the right. The actions are via the 2×2 matrix multiplication. We have

$$(\mu, \Phi) \leftrightarrow \begin{pmatrix} \mu & \Phi_1 & \Phi_2 & \Phi_3 \\ \Phi_1 & \mu & -i\Phi_3 & i\Phi_2 \\ \Phi_2 & i\Phi_3 & \mu & -i\Phi_1 \\ \Phi_3 - i\Phi_2 & i\Phi_1 & \mu & \end{pmatrix} \begin{pmatrix} \bar{\mu} & \bar{\Phi}_1 & \bar{\Phi}_2 & \bar{\Phi}_3 \\ \bar{\Phi}_1 & \bar{\mu} & i\bar{\Phi}_3 - i\bar{\Phi}_2 \\ \bar{\Phi}_2 - i\bar{\Phi}_3 & \bar{\mu} & i\bar{\Phi}_1 \\ \bar{\Phi}_3 & i\bar{\Phi}_2 - i\bar{\Phi}_1 & \bar{\mu} \end{pmatrix} \leftrightarrow (\bar{\mu}, \bar{\Phi}) \quad (\text{B.8})$$

as it follows easily from the definition of these actions. In terms of this representation, an explicit covariant and gauge-invariant form of Maxwell equations is

$$\sum_{a=0}^3 \nabla_a (\mu, \Phi)_b^a = 0 \quad \Leftrightarrow \quad \sum_{a=0}^3 \bar{\nabla}_a (\bar{\mu}, \bar{\Phi})_b^a = 0. \quad (\text{B.9})$$

We can compose the left and right actions, and define for every Φ and Ψ an intrinsic diadic operator field

$$T_{\Phi, \Psi}(\xi) = \Phi \xi \bar{\Psi}. \quad (\text{B.10})$$

It follows from the very definition, that the conjugation ($\Leftrightarrow$ adjoint operation) satisfies

$$T_{\Phi, \Psi}(\xi)^* = T_{\Psi, \Phi}(\xi^*) \quad (\text{B.11})$$

and in particular $T_\Phi = T_{\Phi, \Phi}$ satisfies

$$T_\Phi(\xi)^* = T_\Phi(\xi^*) \quad (\text{B.12})$$

in other words the transformation T_Φ is real. In fact, the electromagnetic energy-momentum tensor is given by

$$\tau = \frac{1}{2} T_\Phi : \xi \mapsto \frac{1}{2} \Phi \xi \bar{\Phi}. \quad (\text{B.13})$$

Indeed, using the matrix representation of Φ and ξ we obtain

$$\begin{aligned} \Phi \xi \bar{\Phi} &= (\Phi_1 \sigma_1 + \Phi_2 \sigma_2 + \Phi_3 \sigma_3)(t + x\sigma_1 + y\sigma_2 + z\sigma_3)(\bar{\Phi}_1 \sigma_1 + \bar{\Phi}_2 \sigma_2 + \bar{\Phi}_3 \sigma_3) \\ &= \left((x, y, z) \cdot \Phi + \left\{ t\Phi + i\Phi \times (x, y, z) \right\} \cdot \vec{\sigma} \right) (\bar{\Phi}_1 \sigma_1 + \bar{\Phi}_2 \sigma_2 + \bar{\Phi}_3 \sigma_3) = t\bar{\Phi} \cdot \Phi \\ &+ ((x, y, z) \cdot \Phi) \bar{\Phi} + it(\Phi \times \bar{\Phi}) - i(\Phi \times \bar{\Phi}) \cdot (x, y, z) + \bar{\Phi} \times (\Phi \times (x, y, z)) = t\bar{\Phi} \cdot \Phi \\ &+ it(\Phi \times \bar{\Phi}) - i(\Phi \times \bar{\Phi}) \cdot (x, y, z) + \left[\Phi \otimes \bar{\Phi} + \bar{\Phi} \otimes \Phi \right] (x, y, z) - (\bar{\Phi} \cdot \Phi)(x, y, z) \end{aligned}$$

which proves that (B.13) is equivalent to our previously encountered expression for the energy-momentum tensor τ_2 . If we expand the domain of our pairings by adding the scalar fields μ besides Φ into the picture, then the corresponding energy-momentum tensor will contain two additional parts, the one represented by the scalar operator $\bar{\mu}\mu/2$ and the second term containing hermitian combinations between μ , $\bar{\mu}$ and Φ , $\bar{\Phi}$. By definition, this is also equivalent to taking 1/2 of the product of the matrices in (B.8). Explicitly $\tau = \tau_0 + \tau_1 + \tau_2$ with

$$\begin{array}{ccc} \tau_0 & \tau_1 & \tau_2 \\ \updownarrow & \updownarrow & \updownarrow \\ \begin{pmatrix} \varpi & 0 & 0 & 0 \\ 0 & \varpi & 0 & 0 \\ 0 & 0 & \varpi & 0 \\ 0 & 0 & 0 & \varpi \end{pmatrix} & \begin{pmatrix} 0 & \vec{p}_1 & \vec{p}_2 & \vec{p}_3 \\ \vec{p}_1 & 0 & -\vec{q}_3 & \vec{q}_2 \\ \vec{p}_2 & \vec{q}_3 & 0 & -\vec{q}_1 \\ \vec{p}_3 & -\vec{q}_2 & \vec{q}_1 & 0 \end{pmatrix} & \begin{pmatrix} w & -\mathbb{P}_1 & -\mathbb{P}_2 & -\mathbb{P}_3 \\ \mathbb{P}_1 & \mathbb{M}_{11} & \mathbb{M}_{12} & \mathbb{M}_{13} \\ \mathbb{P}_2 & \mathbb{M}_{21} & \mathbb{M}_{22} & \mathbb{M}_{23} \\ \mathbb{P}_3 & \mathbb{M}_{31} & \mathbb{M}_{32} & \mathbb{M}_{33} \end{pmatrix} \end{array}$$

and we have represented the tensor in natural space-time coordinates where the metric tensor is $\text{diag}(1, -1, -1, -1)$.

Since all the basic entities of the theory are interpretable in the intrinsic way, coming from the geometry of the Minkowski space-time, the whole geometrical context is intrinsically Poincare-invariant.

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ON THE $W^\pm$ AND Z^0 MASSES

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Abstract

Scalar and vector fields are coupled in a gauge invariant manner, such as to form massive vector fields. In this, there is no condensate or vacuum expectation value. Transverse and longitudinal solutions are found for the $W^\pm$ and Z^0 bosons. They satisfy the nonlinear cubic wave equation. Total energy and momentum are calculated, and this determines the mass ratio m_W/m_Z .

1 Introduction

In the electroweak theory, as it is currently understood, a uniform condensate is imposed upon the system of coupled scalar and vector fields. This gives rise to the linear wave equation for massive bosons. However, there is no experimental evidence for such a condensate, and it would be desirable to eliminate it altogether. In the theory presented here, the scalar-vector system yields the nonlinear cubic wave equation. It does so in a straightforward way, without resorting to unphysical assumptions. Vector boson solutions are found which have well-defined mass.

Historically, the U(1) model of scalar electrodynamics was used to create massive photons [1]. However, it was only with the advent of electroweak theory that the weak constants, g and g' , were thought to play a similar role. In this paper, the weak coupling between scalar and vector fields is shown to generate mass. The coupling terms occur in the standard electroweak Lagrangian. Transverse and longitudinal solutions are found for the $W^\pm$ and Z^0 bosons. Energy and momentum are shown to be conserved, and the total energy and momentum are calculated. The mass ratio m_W/m_Z emerges, during the course of this calculation.

The paper begins by revisiting (and revising) the U(1) model. Most of the analysis carries over to the electroweak theory.

2 U(1): equations of motion

The U(1) Lagrangian is [2]

$$L = L(A) + L(\Phi) + L(k) \quad (1)$$

where

$$L(A) = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \quad (2)$$

$$L(\Phi) = g^{\mu\nu}(D_\mu\Phi)^*(D_\nu\Phi) \quad (3)$$

$$L(k) = -\frac{k^2}{6g^2}g^{\mu\nu}k_\mu k_\nu \quad (4)$$

The constant term $L(k)$ does not enter the field equations, but it will contribute to the energy tensor. The U(1) covariant derivatives are

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad (5)$$

$$D_\mu \Phi = \partial_\mu \Phi + ig A_\mu \Phi \quad (6)$$

The functional derivatives

$$\frac{\partial L}{\partial(\partial_\mu A_\nu)} = -F^{\mu\nu} \quad (7)$$

$$\frac{\partial L}{\partial A_\nu} = 2g^2 A^\nu \Phi^* \Phi + ig\{(\partial^\nu \Phi^*)\Phi - \Phi^* \partial^\nu \Phi\} \quad (8)$$

$$\frac{\partial L}{\partial(\partial_\mu \Phi^*)} = \partial^\mu \Phi + ig A^\mu \Phi \quad (9)$$

$$\frac{\partial L}{\partial \Phi^*} = g^2 A_\mu A^\mu \Phi - ig A^\mu \partial_\mu \Phi \quad (10)$$

yield coupled equations of motion

$$-\partial_\mu F^{\mu\nu} = 2g^2 A^\nu \Phi^* \Phi + ig\{(\partial^\nu \Phi^*)\Phi - \Phi^* \partial^\nu \Phi\} \quad (11)$$

and

$$\partial_\mu \partial^\mu \Phi = g^2 A_\mu A^\mu \Phi - ig\{(\partial_\mu A^\mu)\Phi + 2A^\mu \partial_\mu \Phi\} \quad (12)$$

The U(1) gauge invariance allows the transformation

$$\Phi = \frac{1}{\sqrt{2}}(\phi + i\psi) \longrightarrow \frac{\phi}{\sqrt{2}} \quad (13)$$

where $\phi(x)$ is a real function [3]. In this unitary gauge, equations (11) and (12) are simplified

$$\partial_\mu F^{\mu\nu} + g^2 A^\nu \phi^2 = 0 \quad (14)$$

$$\partial_\mu \partial^\mu \phi - g^2 A_\mu A^\mu \phi = 0 \quad (15)$$

$$(\partial_\mu A^\mu)\phi + 2A^\mu \partial_\mu \phi = 0 \quad (16)$$

Since $\partial_\mu \partial_\nu F^{\mu\nu} \equiv 0$, it follows from (14) that

$$\partial_\mu (A^\mu \phi^2) = 0 \quad (17)$$

which agrees with (16). These equations admit traveling solutions, if A^μ is a polarized vector. In this case,

$$A^\mu(u) = \epsilon^\mu f(u) \quad (18)$$

where the argument $u = -k_\mu x^\mu$, and ϵ^μ is a real polarization vector.¹ Such vectors satisfy

$$\partial_\mu A^\mu(u) = -k_\mu \epsilon^\mu \frac{df(u)}{du} = 0 \quad (19)$$

since $k_\mu \epsilon^\mu = 0$. From (16), it must also be true that

$$A^\mu \partial_\mu \phi = 0 \quad (20)$$

This condition is satisfied, if $\phi = \phi(u)$

$$A^\mu \partial_\mu \phi = -k_\mu \epsilon^\mu f(u) \frac{d\phi}{du} = 0 \quad (21)$$

Polarization vectors are space-like, $\epsilon_\mu \epsilon^\mu = -1$, so that

$$A_\mu A^\mu = -f^2(u) \quad (22)$$

Therefore, both equations (14) and (15) will be satisfied, if $f(u) = \phi(u)$ and

$$\partial_\mu \partial^\mu \phi(u) + g^2 \phi^3(u) = 0 \quad (23)$$

This equation, known as the cubic wave equation, is solved by the elliptic function $\text{cn}(u, \frac{1}{2})$ (appendix A)

$$\phi(u) = \frac{k}{g} \text{cn}(-k_\mu x^\mu) \quad (24)$$

The amplitude of this traveling wave is not arbitrary, but is fixed by the value of k/g . It will be shown in the following section that k is directly proportional to the mass.²

¹Throughout this paper, $u = -k_\mu x^\mu = (\mathbf{k} \cdot \mathbf{x} - k^0 x^0)$ and $k^2 = k_\mu k^\mu = (k^0)^2 - \mathbf{k}^2$.

²The condition $f(u) = \phi(u)$ removes one degree of freedom from the system, leaving three. They belong to the massive vector field. The scalar field is no longer independent, since $\phi(u) = -\epsilon_\mu A^\mu(u)$.

3 $U(1)$: energy and momentum

The energy tensor for the vector field is

$$T_{\mu\nu}(A) = F_{\mu\eta}F_{\nu}^{\eta} + \frac{1}{4}g_{\mu\nu}F_{\eta\rho}F^{\eta\rho} \quad (25)$$

with energy and momentum densities

$$T_{00}(A) = \frac{1}{2}\{F_{01}^2 + F_{02}^2 + F_{03}^2 + F_{23}^2 + F_{31}^2 + F_{12}^2\} \quad (26)$$

$$T_{0i}(A) = F_{01}F_{i1} + F_{02}F_{i2} + F_{03}F_{i3} \quad (27)$$

For the real scalar field

$$T_{\mu\nu}(\phi) = \partial_{\mu}\phi\partial_{\nu}\phi + g^2A_{\mu}A_{\nu}\phi^2 - g_{\mu\nu}L(\phi) \quad (28)$$

with energy and momentum densities

$$T_{00}(\phi) = \frac{1}{2}\{(\partial_0\phi)^2 + (\nabla\phi)^2 + g^2[(A_0)^2 + \mathbf{A}^2]\phi^2\} \quad (29)$$

$$T_{0i}(\phi) = \partial_0\phi\partial_i\phi + g^2A_0A_i\phi^2 \quad (30)$$

The constant term $L(k)$ contributes

$$T_{\mu\nu}(k) = -\frac{k^2}{3g^2}(k_{\mu}k_{\nu} - \frac{1}{2}g_{\mu\nu}k^2) \quad (31)$$

so that

$$T_{00}(k) = -\frac{k^2}{6g^2}(k_0^2 + \mathbf{k}^2) \quad (32)$$

$$T_{0i}(k) = -\frac{k^2}{3g^2}k_0k_i \quad (33)$$

The following formulas occur repeatedly in the calculations and are placed here for reference (app. A):

$$\phi^2 = \frac{k^2}{g^2}\text{cn}^2(u) \quad (34)$$

$$\left(\frac{d\phi}{du}\right)^2 = \frac{k^2}{2g^2}\{1 - \text{cn}^4(u)\} \quad (35)$$

3.1 Transverse polarization

For plane waves moving along the x^3 -axis, $k^\mu = (k^0, k^3)$. The linear polarization vectors are

$$\epsilon_1^\mu = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \epsilon_2^\mu = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad (36)$$

If the polarization is along x^1 , then $A^\mu = \epsilon_1^\mu \phi(u)$ has the single component $A^1 = \phi(u)$. In this case, the energy contributions are

$$\begin{aligned} T_{00}(A) &= \frac{1}{2}(\partial_0 A_1)^2 + \frac{1}{2}(\partial_3 A_1)^2 = \frac{k_0^2 + k_3^2}{2} \left(\frac{d\phi}{du} \right)^2 \\ &= \frac{k^2}{2g^2} \frac{k_0^2 + k_3^2}{2} \{1 - \text{cn}^4(u)\} \end{aligned} \quad (37)$$

$$\begin{aligned} T_{00}(\phi) &= \frac{1}{2} \{ (\partial_0 \phi)^2 + (\partial_3 \phi)^2 + g^2 A_1^2 \phi^2 \} \\ &= \frac{k^2}{2g^2} \left\{ \frac{k_0^2 + k_3^2}{2} \{1 - \text{cn}^4(u)\} + (k_0^2 - k_3^2) \text{cn}^4(u) \right\} \end{aligned} \quad (38)$$

$$T_{00}(k) = -\frac{k^2}{2g^2} \frac{k_0^2 + k_3^2}{3} \quad (39)$$

Similarly, the momenta are

$$T_{03}(A) = \partial_0 A_1 \partial_3 A_1 = \frac{k^2}{2g^2} k_0 k_3 \{1 - \text{cn}^4(u)\} \quad (40)$$

$$T_{03}(\phi) = \partial_0 \phi \partial_3 \phi = \frac{k^2}{2g^2} k_0 k_3 \{1 - \text{cn}^4(u)\} \quad (41)$$

$$T_{03}(k) = -\frac{k^2}{2g^2} \frac{2}{3} k_0 k_3 \quad (42)$$

Sum terms, in order to obtain

$$T_{00} = \frac{k^2}{2g^2} \left\{ \frac{2}{3} (k_0^2 + k_3^2) - 2k_3^2 \text{cn}^4(u) \right\} \quad (43)$$

$$T_{03} = \frac{k^2}{2g^2} k_0 k_3 \left\{ \frac{4}{3} - 2 \text{cn}^4(u) \right\} \quad (44)$$

These expressions obey the energy conservation law, $\partial_\nu T^{0\nu} = 0$. It can be shown that the momentum is conserved as well, $\partial_\nu T^{3\nu} = 0$.

The calculation of total energy and momentum follows the introduction of a volume element $l^3 dV/V$. In order to arrive at the correct quantum expressions, the integrals must be independent of the ratio k^2/g^2 . This factor is eliminated from (43) and (44) by setting

$$\frac{l^3}{V} dV = \frac{3\pi}{2K} \frac{g^2}{k^2 k^0 V} dV \quad (45)$$

The integrals are

$$\begin{aligned} E &= \frac{3\pi}{2K} \frac{g^2}{k^2 k^0 V} \int T_0^0 dV \\ &= \frac{3\pi}{2K} \frac{\hbar c}{2k^0 V} \int \left\{ \frac{2}{3}(k_0^2 + k_3^2) - 2k_3^2 \text{cn}^4(u) \right\} dV \end{aligned} \quad (46)$$

$$\begin{aligned} cp^3 &= \frac{3\pi}{2K} \frac{g^2}{k^2 k^0 V} \int T_0^3 dV \\ &= \frac{3\pi}{2K} \frac{\hbar c}{2k^0 V} \int k_0 k^3 \left\{ \frac{4}{3} - 2 \text{cn}^4(u) \right\} dV \end{aligned} \quad (47)$$

The elliptic function $\text{cn}(u, \frac{1}{2})$ admits the integral (app. A)

$$\int \text{cn}^4(u) du = \frac{u}{3} + \frac{2}{3} \text{sn}(u) \text{cn}(u) \text{dn}(u) + \text{constant} \quad (48)$$

Over one period, (or any integral number of periods)

$$\frac{1}{4K} \int_0^{4K} \text{cn}^4(u) du = \frac{1}{3} \quad (49)$$

$\text{sn}(0) = \text{sn}(4K) = 0$. Therefore, the energy and momentum are given by

$$E = \frac{\pi}{2K} \hbar c k^0 = \hbar \omega \quad (50)$$

$$cp^3 = \frac{\pi}{2K} \hbar c k^3 = \hbar c \frac{2\pi}{\lambda} \quad (51)$$

3.2 Longitudinal polarization

In this case, $A^\mu(u) = \epsilon^\mu \phi(u)$ is expressed in terms of the zero-helicity vector

$$\epsilon^\mu(0) = \frac{1}{k} \begin{pmatrix} k^3 \\ 0 \\ 0 \\ k^0 \end{pmatrix} \quad (52)$$

A^μ has two components $(A^0, A^3) = (k^3, k^0)k^{-1}\phi(u)$. The contributions to the energy density are

$$T_{00}(A) = \frac{1}{2}(\partial_0 A_3 - \partial_3 A_0)^2 = \frac{k^2}{2g^2} \frac{k_0^2 - k_3^2}{2} \{1 - \text{cn}^4(u)\} \quad (53)$$

$$\begin{aligned} T_{00}(\phi) &= \frac{1}{2} \{ (\partial_0 \phi)^2 + (\partial_3 \phi)^2 + g^2 (A_0^2 + A_3^2) \phi^2 \} \\ &= \frac{k^2}{2g^2} \frac{k_0^2 + k_3^2}{2} \{1 + \text{cn}^4(u)\} \end{aligned} \quad (54)$$

$$T_{00}(k) = -\frac{k^2}{2g^2} \frac{k_0^2 + k_3^2}{3} \quad (55)$$

while those for the momentum are

$$T_{03}(A) = 0 \quad (56)$$

$$T_{03}(\phi) = \partial_0 \phi \partial_3 \phi + g^2 A_0 A_3 \phi^2 = \frac{k^2}{2g^2} k_0 k_3 \{1 + \text{cn}^4(u)\} \quad (57)$$

$$T_{03}(k) = -\frac{k^2}{2g^2} \frac{2}{3} k_0 k_3 \quad (58)$$

Sum terms to find

$$T_{00} = \frac{k^2}{2g^2} \left\{ \frac{2}{3} k_0^2 - \frac{1}{3} k_3^2 + k_3^2 \text{cn}^4(u) \right\} \quad (59)$$

$$T_{03} = \frac{k^2}{2g^2} k_0 k_3 \left\{ \frac{1}{3} + \text{cn}^4(u) \right\} \quad (60)$$

Here, too, the energy and momentum are conserved, $\partial_\nu T^{0\nu} = \partial_\nu T^{3\nu} = 0$. The integrals are

$$\begin{aligned} E &= \frac{3\pi}{2K} \frac{g^2}{k^2 k^0 V} \int T_0^0 dV \\ &= \frac{3\pi}{2K} \frac{\hbar c}{2k^0 V} \int \left\{ \frac{2}{3} k_0^2 - \frac{1}{3} k_3^2 + k_3^2 \text{cn}^4(u) \right\} dV \\ &= \frac{\pi}{2K} \hbar c k^0 = \hbar \omega \end{aligned} \quad (61)$$

$$\begin{aligned} cp^3 &= \frac{3\pi}{2K} \frac{g^2}{k^2 k^0 V} \int T_0^3 dV \\ &= \frac{3\pi}{2K} \frac{\hbar c}{2k^0 V} \int k_0 k^3 \left\{ \frac{1}{3} + \text{cn}^4(u) \right\} dV \\ &= \frac{\pi}{2K} \hbar c k^3 = \hbar c \frac{2\pi}{\lambda} \end{aligned} \quad (62)$$

These calculations establish the equality

$$E^2 - \mathbf{p}^2 = \left(\frac{\pi}{2K} \right)^2 k^2 \quad (63)$$

showing that k is directly proportional to the mass

$$m = \frac{\pi}{2K} k \quad (64)$$

4 $U(1) \otimes SU(2)_L$: equations of motion; energy tensors

The Lagrangian is given by

$$L = L(W) + L(B) + L(\Phi) + L(k) + L(\text{leptons}) \quad (65)$$

The lepton term contains the electroweak interaction and will be set aside. The focus is upon the coupling between vector and scalar fields. The covariant derivatives are [4]

$$G_{\mu\nu}(W) = \partial_\mu(W_\nu^i T^i) - \partial_\nu(W_\mu^i T^i) + g \Sigma_{ijk} \epsilon_{ijk} W_\mu^i W_\nu^j T^k \quad (66)$$

$$F_{\mu\nu}(B) = \partial_\mu B_\nu - \partial_\nu B_\mu \quad (67)$$

$$\begin{aligned} D_\mu \Phi &= \left\{ \partial_\mu + ig T^i W_\mu^i + ig' \frac{Y}{2} B_\mu \right\} \Phi \\ &= \left\{ \partial_\mu + ig \frac{\tau^i}{2} W_\mu^i + ig' \frac{1}{2} B_\mu \right\} \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} \end{aligned} \quad (68)$$

The final term in $G_{\mu\nu}(W)$ yields third- and fourth-order vector boson interactions and will also be ignored [5]. This leaves

$$\begin{aligned} L(W) + L(B) &= -\frac{1}{2}\text{Tr } G_{\mu\nu}(W)G^{\mu\nu}(W) - \frac{1}{4}F_{\mu\nu}(B)F^{\mu\nu}(B) \\ &= -\frac{1}{4}\left\{F_{\mu\nu}(A)F^{\mu\nu}(A) + F_{\mu\nu}(Z)F^{\mu\nu}(Z) \right. \\ &\quad \left. + F_{\mu\nu}(W^1)F^{\mu\nu}(W^1) + F_{\mu\nu}(W^2)F^{\mu\nu}(W^2)\right\} \end{aligned} \quad (69)$$

where

$$W_\mu^3 = \frac{g'}{\sqrt{g^2 + g'^2}}A_\mu + \frac{g}{\sqrt{g^2 + g'^2}}Z_\mu = \sin\theta A_\mu + \cos\theta Z_\mu \quad (70)$$

$$B_\mu = \frac{g}{\sqrt{g^2 + g'^2}}A_\mu - \frac{g'}{\sqrt{g^2 + g'^2}}Z_\mu = \cos\theta A_\mu - \sin\theta Z_\mu \quad (71)$$

The charged bosons $W_\mu^\pm = (W_\mu^1 \mp iW_\mu^2)/\sqrt{2}$ are expressed in terms of the real fields W_μ^1 and W_μ^2 .

The expansion of

$$L(\Phi) = g^{\mu\nu}(D_\mu\Phi)^\dagger D_\nu\Phi \quad (72)$$

is carried out in appendix B, where the functional derivatives are also found. Before writing the equations of motion, the scalar field is transformed to unitary gauge

$$\Phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 \\ \phi/\sqrt{2} \end{pmatrix} \quad (73)$$

where $\phi(x)$ is real. This greatly simplifies the functional derivatives

$$\frac{\partial L}{\partial(\partial_\mu A_\nu)} = -F^{\mu\nu}(A) \quad \frac{\partial L}{\partial A_\nu} = 0 \quad (74)$$

$$\frac{\partial L}{\partial(\partial_\mu Z_\nu)} = -F^{\mu\nu}(Z) \quad \frac{\partial L}{\partial Z_\nu} = \frac{1}{4}(g^2 + g'^2)Z^\nu\phi^2 \quad (75)$$

$$\frac{\partial L}{\partial(\partial_\mu W_\nu)} = -F^{\mu\nu}(W) \quad \frac{\partial L}{\partial W_\nu} = \frac{1}{4}g^2W^\nu\phi^2 \quad (76)$$

where W_ν is either W_ν^1 or W_ν^2 . The equations of motion for the vector fields are

$$-\partial_\mu F^{\mu\nu}(A) = 0 \quad (77)$$

$$-\partial_\mu F^{\mu\nu}(Z) = \frac{1}{4}(g^2 + g'^2)Z^\nu\phi^2 \quad (78)$$

$$-\partial_\mu F^{\mu\nu}(W) = \frac{1}{4}g^2 W^\nu\phi^2 \quad (79)$$

Since $\partial_\mu\partial_\nu F^{\mu\nu} \equiv 0$, these equations yield

$$\partial_\nu(Z^\nu\phi^2) = \partial_\nu(W^\nu\phi^2) = 0 \quad (80)$$

For the scalar field,

$$\frac{\partial L}{\partial(\partial_\mu\phi^{0*})} = \partial^\mu\phi^0 - \frac{i}{2}\sqrt{g^2 + g'^2} Z^\mu\phi^0 \quad (81)$$

$$\frac{\partial L}{\partial\phi^{0*}} = \frac{i}{2}\sqrt{g^2 + g'^2} Z^\mu\partial_\mu\phi^0 + \frac{g^2}{2} W^{-\mu}W_\mu^+\phi^0 + \frac{1}{4}(g^2 + g'^2)Z^\mu Z_\mu\phi^0 \quad (82)$$

$$\frac{\partial L}{\partial(\partial_\mu\phi^{+*})} = \frac{i}{\sqrt{2}} gW^{+\mu}\phi^0 \quad (83)$$

$$\frac{\partial L}{\partial\phi^{+*}} = -\frac{i}{\sqrt{2}} gW^{+\mu}\partial_\mu\phi^0 + (eA^\mu - g'\sin\theta Z^\mu)\frac{1}{\sqrt{2}}gW_\mu^+\phi^0 \quad (84)$$

The equations of motion are ($\phi^0 = \phi/\sqrt{2}$)

$$\partial_\mu\partial^\mu\phi - \frac{1}{4}g^2 W^{1\mu}W_\mu^1\phi - \frac{1}{4}g^2 W^{2\mu}W_\mu^2\phi - \frac{1}{4}(g^2 + g'^2)Z^\mu Z_\mu\phi = 0 \quad (85)$$

and

$$(eA^\mu - g'\sin\theta Z^\mu) g W_\mu^+\phi = 0 \quad (86)$$

where (80) has been used. The vector fields couple individually with ϕ , so that (86) will be satisfied.

The energy tensors are much the same as for U(1). Each vector field A_μ , W_μ^1 , W_μ^2 , Z_μ contributes

$$T_{\mu\nu} = F_{\mu\eta}F_{\nu}^{\eta} + \frac{1}{4}g_{\mu\nu}F_{\eta\rho}F^{\eta\rho} \quad (87)$$

while the scalar Lagrangian

$$L(\phi) = \frac{1}{2}g^{\mu\nu}\left\{\partial_{\mu}\phi\partial_{\nu}\phi + \frac{1}{4}g^2(W_{\mu}^1W_{\nu}^1 + W_{\mu}^2W_{\nu}^2)\phi^2 + \frac{1}{4}(g^2 + g'^2)Z_{\mu}Z_{\nu}\phi^2\right\} \quad (88)$$

gives

$$T_{\mu\nu}(\phi) = \partial_{\mu}\phi\partial_{\nu}\phi + \frac{1}{4}g^2(W_{\mu}^1W_{\nu}^1 + W_{\mu}^2W_{\nu}^2)\phi^2 + \frac{1}{4}(g^2 + g'^2)Z_{\mu}Z_{\nu}\phi^2 - g_{\mu\nu}L(\phi) \quad (89)$$

The constant term $T_{\mu\nu}(k)$ is treated in the following section.

5 The $W^{\pm}$ and Z^0 bosons

The coupling between $W^{\pm}$ and ϕ is described by (79, 85)

$$\partial_{\mu}F^{\mu\nu}(W) + \frac{1}{4}g^2W^{\nu}\phi^2 = 0 \quad (90)$$

$$\partial_{\mu}\partial^{\mu}\phi - \frac{1}{4}g^2W_{\mu}W^{\mu}\phi = 0 \quad (91)$$

where $\partial_{\mu}W^{\mu} = W^{\mu}\partial_{\mu}\phi = 0$. These equations are identical to those of U(1), (14) and (15), with the replacement $g \rightarrow g/2$. The solutions of the cubic wave equation (23) are polarized, $W^{\mu}(u) = \epsilon^{\mu}\phi(u)$, where

$$\phi(u) = \frac{2k_W}{g} \text{cn}(-k_{\mu}x^{\mu}) \quad (92)$$

Similarly, the coupling between Z^0 and ϕ is given by

$$\partial_{\mu}F^{\mu\nu}(Z) + \frac{1}{4}(g^2 + g'^2)Z^{\nu}\phi^2 = 0 \quad (93)$$

$$\partial_{\mu}\partial^{\mu}\phi - \frac{1}{4}(g^2 + g'^2)Z_{\mu}Z^{\mu}\phi = 0 \quad (94)$$

The solutions are $Z^{\mu}(u) = \epsilon^{\mu}\phi(u)$, where

$$\phi(u) = \frac{2k_Z}{\sqrt{g^2 + g'^2}} \text{cn}(-k_\mu x^\mu) \quad (95)$$

There are eleven degrees of freedom in the system, belonging to the massless photon and the three massive bosons. The scalar field is not independent, i.e., $\phi(u) = -\epsilon_\mu W^\mu(u)$ or $\phi(u) = -\epsilon_\mu Z^\mu(u)$.

The expressions for T_{00} and T_{03} are exactly the same as those in U(1), apart from the new coupling constants. They give rise to factors k_W^2/g^2 and $k_Z^2/(g^2 + g'^2)$. Before integration can occur, the volume element must eliminate these two factors. This is possible, if they are equal

$$\frac{k_W^2}{g^2} = \frac{k_Z^2}{g^2 + g'^2} \quad (96)$$

Only then will the volume element be uniquely defined (compare (45))

$$\frac{l^3}{V} dV = \frac{3\pi}{8K} \frac{g^2}{k_W^2 k^0 V} dV = \frac{3\pi}{8K} \frac{(g^2 + g'^2)}{k_Z^2 k^0 V} dV \quad (97)$$

Integration may now go forward as in U(1).³ The integrals for $W^\pm$ and Z^0 , including transverse and longitudinal fields, all assume the form

$$E = \frac{l^3}{V} \int T_0^0 dV = \hbar\omega \quad (98)$$

$$cp^3 = \frac{l^3}{V} \int T_0^3 dV = \hbar c \frac{2\pi}{\lambda} \quad (99)$$

6 Concluding remarks

The coupling theory is made tractable with the choice of unitary gauge. It eliminates the scalar currents, leaving direct nonlinear coupling between the real scalar and vector fields. The equations of motion yield both transverse and longitudinal solutions, indicating the presence of mass.

³Moreover, the constant terms are identical

$$\begin{aligned} T_{\mu\nu}(k) &= -\frac{4}{3} \frac{k_W^2}{g^2} (k_\mu k_\nu - \frac{1}{2} g_{\mu\nu} k^2) \\ &= -\frac{4}{3} \frac{k_Z^2}{g^2 + g'^2} (k_\mu k_\nu - \frac{1}{2} g_{\mu\nu} k^2) \end{aligned}$$

In the U(1) model, scalar-vector coupling generates a single massive vector field, while in the electroweak theory there are three, the $W^\pm$ and Z^0 . They satisfy relation (96), yielding the mass ratio

$$\frac{m_W}{m_Z} = \frac{k_W}{k_Z} = \frac{g}{\sqrt{g^2 + g'^2}} \quad (100)$$

In linear field theory, it is customary to include the integration volume V as part of the plane wave expansion. For example, the expression $(\hbar c/2k^0 V)^{1/2}$ appears in each boson term. This is possible because the energy tensor $T_{\mu\nu}$ is second-order in the field. However, the non-linear tensor (89) contains both second- and fourth-order terms, making such a procedure untenable. Instead, the volume is introduced as a factor, l^3/V , during the integration of $T_{\mu\nu}$.

Finally, the constant term in the energy tensor $T_{\mu\nu}(k)$ has been introduced out of necessity. Without it, the energy and momentum would not satisfy the relation $c^2 \mathbf{p} = E \mathbf{v}$, as they must; the work would simply be incomplete.

Appendix A: Elliptic functions [6, 7]

The elliptic functions $\text{sn}(u, m)$, $\text{cn}(u, m)$, and $\text{dn}(u, m)$ satisfy

$$\text{sn}^2(u, m) + \text{cn}^2(u, m) = 1 \quad (101)$$

$$\text{dn}(u, m) = \sqrt{1 - m \text{sn}^2(u, m)} \quad (102)$$

Their derivatives are

$$\frac{d \text{sn}(u, m)}{du} = \text{cn}(u, m) \text{dn}(u, m) \quad (103)$$

$$\frac{d \text{cn}(u, m)}{du} = -\text{sn}(u, m) \text{dn}(u, m) \quad (104)$$

$$\frac{d \text{dn}(u, m)}{du} = -m \text{sn}(u, m) \text{cn}(u, m) \quad (105)$$

In particular,

$$\begin{aligned} \frac{d^2 \text{cn}(u, m)}{du^2} &= -\text{cn}(u) \text{dn}^2(u) + m \text{sn}^2(u) \text{cn}(u) \\ &= (2m - 1) \text{cn}(u) - 2m \text{cn}^3(u) \end{aligned} \quad (106)$$

If the parameter $m = \frac{1}{2}$, then

$$\frac{d^2 \text{cn}(u, \frac{1}{2})}{du^2} = -\text{cn}^3(u, \frac{1}{2}) \quad (107)$$

The cubic wave equation (23) yields the ordinary differential equation

$$k_\mu k^\mu \frac{d^2 \phi(u)}{du^2} + g^2 \phi^3(u) = 0 \quad (108)$$

Substitute $\phi(u) = a \text{cn}(u, \frac{1}{2})$ to find

$$-k^2 a \text{cn}^3(u, \frac{1}{2}) + g^2 a^3 \text{cn}^3(u, \frac{1}{2}) = 0 \quad (109)$$

This equation is satisfied, if $a^2 = k^2/g^2$. The solution of (108) is

$$\phi(u) = \frac{k}{g} \text{cn}(u, \frac{1}{2}) = \frac{k}{g} \text{cn}(-k_\mu x^\mu) \quad (110)$$

The elliptic functions, with $m = \frac{1}{2}$, satisfy two useful identities. The first is

$$\left(\frac{d \text{cn}(u)}{du}\right)^2 = \text{sn}^2(u) \text{dn}^2(u) = \frac{1}{2} \{1 - \text{cn}^4(u)\} \quad (111)$$

The second identity is

$$\frac{d}{du} \{\text{sn}(u) \text{cn}(u) \text{dn}(u)\} = \frac{1}{2} \{3 \text{cn}^4(u) - 1\} \quad (112)$$

and it follows that

$$\int \text{cn}^4(u) du = \frac{u}{3} + \frac{2}{3} \text{sn}(u) \text{cn}(u) \text{dn}(u) + \text{constant} \quad (113)$$

The period of an elliptic function is $4K$. (If $m = \frac{1}{2}$, then $K \doteq 1.85$.) For motion along the x^3 -axis,

$$\text{cn}(-k_\mu x^\mu) = \text{cn}(k^3 x^3 - k^0 x^0) = \text{cn} 4K \left(\frac{x^3}{\lambda} - \frac{t}{T} \right) \quad (114)$$

It follows that

$$ck^0 = \frac{4K}{T} = \frac{2K}{\pi} 2\pi f = \frac{2K}{\pi} \omega \quad (115)$$

$$k^3 = \frac{4K}{\lambda} = \frac{2K}{\pi} \frac{2\pi}{\lambda} \quad (116)$$

and

$$E = \hbar\omega = \frac{\pi}{2K} \hbar ck^0 \quad (117)$$

$$cp^3 = \hbar c \frac{2\pi}{\lambda} = \frac{\pi}{2K} \hbar ck^3 \quad (118)$$

Appendix B: Scalar field

B.1 Lagrangian

$$\begin{aligned} L(\Phi) &= g^{\mu\nu} (D_\mu \Phi)^\dagger (D_\nu \Phi) \\ &= \partial^\mu \phi^{+*} \partial_\mu \phi^+ + \partial^\mu \phi^{0*} \partial_\mu \phi^0 \\ &\quad + \left\{ [eA^\mu + \frac{1}{2}(g \cos \theta - g' \sin \theta) Z^\mu]^2 + \frac{1}{2} g^2 W^{+\mu} W_\mu^- \right\} \phi^{+*} \phi^+ \\ &\quad + [eA^\mu - g' \sin \theta Z^\mu] \left\{ \frac{g}{\sqrt{2}} W_\mu^+ \phi^{+*} \phi^0 + \frac{g}{\sqrt{2}} W_\mu^- \phi^{0*} \phi^+ \right\} \\ &\quad + \left\{ \frac{1}{2} g^2 W^{-\mu} W_\mu^+ + \frac{1}{4} (g^2 + g'^2) Z^\mu Z_\mu \right\} \phi^{0*} \phi^0 \\ &\quad - i [eA^\mu + \frac{1}{2}(g \cos \theta - g' \sin \theta) Z^\mu] (\phi^{+*} \partial_\mu \phi^+ - \partial_\mu \phi^{+*} \phi^+) \\ &\quad - \frac{i}{\sqrt{2}} g W^{-\mu} (\phi^{0*} \partial_\mu \phi^+ - \partial_\mu \phi^{0*} \phi^+) - \frac{i}{\sqrt{2}} g W^{+\mu} (\phi^{+*} \partial_\mu \phi^0 - \partial_\mu \phi^{+*} \phi^0) \\ &\quad + \frac{i}{2} \sqrt{g^2 + g'^2} Z^\mu (\phi^{0*} \partial_\mu \phi^0 - \partial_\mu \phi^{0*} \phi^0) \end{aligned} \quad (119)$$

B.2 Functional derivatives

$$\begin{aligned} \frac{\partial L}{\partial (\partial_\mu \phi^{0*})} &= \partial^\mu \phi^0 + \frac{i}{\sqrt{2}} g W^{-\mu} \phi^+ - \frac{i}{2} \sqrt{g^2 + g'^2} Z^\mu \phi^0 \\ \frac{\partial L}{\partial \phi^{0*}} &= -\frac{i}{\sqrt{2}} g W^{-\mu} \partial_\mu \phi^+ + \frac{i}{2} \sqrt{g^2 + g'^2} Z^\mu \partial_\mu \phi^0 \end{aligned} \quad (120)$$

$$\begin{aligned}
 & +[eA^\mu - g' \sin \theta Z^\mu] \frac{1}{\sqrt{2}} g W_\mu^- \phi^+ \\
 & + \frac{1}{2} g^2 W^{-\mu} W_\mu^+ \phi^0 + \frac{1}{4} (g^2 + g'^2) Z^\mu Z_\mu \phi^0
 \end{aligned} \tag{121}$$

and

$$\frac{\partial L}{\partial (\partial_\mu \phi^{+*})} = \partial^\mu \phi^+ + i[eA^\mu + \frac{1}{2}(g \cos \theta - g' \sin \theta) Z^\mu] \phi^+ + \frac{i}{\sqrt{2}} g W^{+\mu} \phi^0 \tag{122}$$

$$\begin{aligned}
 \frac{\partial L}{\partial \phi^{+*}} = & -i[eA^\mu + \frac{1}{2}(g \cos \theta - g' \sin \theta) Z^\mu] \partial_\mu \phi^+ + \frac{1}{2} g^2 W^{+\mu} W_\mu^- \phi^+ \\
 & + [eA^\mu + \frac{1}{2}(g \cos \theta - g' \sin \theta) Z^\mu]^2 \phi^+ \\
 & + [eA^\mu - g' \sin \theta Z^\mu] \frac{1}{\sqrt{2}} g W_\mu^+ \phi^0 - \frac{i}{\sqrt{2}} g W^{+\mu} \partial_\mu \phi^0
 \end{aligned} \tag{123}$$

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**ON SOME CHARACTERIZATIONS OF RULED SURFACE OF
A CLOSED SPACELIKE CURVE WITH TIMELIKE
BINORMAL IN DUAL LORENTZIAN SPACE**

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Abstract

A ruled surface is a surface that can be swept out by moving a line in space. Ruled surfaces may be generated or defined in several ways. In this paper, we define the closed ruled surface which generated by a spacelike curve with timelike binormal. Then we calculate the pitch, the angle of pitch and drall of parallel ruled surface. More over we describe parallel ruled surface which produced by a spacelike curve with timelike binormal. We investigate the relations between the pitch length, the dual angle of pitch and drall of parallel ruled surface of a closed spacelike curve with timelike binormal in dual Lorentzian space. Finally we compute the pitch length, the dual angle of pitch and drall of the ruled surface which generated by a unit vectors $\vec{C}$ and $\vec{\bar{C}}$ in the instantaneous dual Darboux vectors $\vec{\bar{\psi}}$ and $\vec{\psi}$.

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1 Introduction

A ruled surface in differential geometry is a surface formed by a motion of a straight line. The lines that belongs to this surface are called generators, and every curve that intersects all the generators is called a directrix. Ruled surfaces are momentous in spatial geometry.

Ruled surfaces have been used in forming cars, ships, processing of products and many other areas such as motion analysis and simulation of rigid body and model-based object recognition sytems. Modern surface modeling systems include ruled surfaces. The geometry of ruded surfaces is basic to labour kinematical and circumstantial mechanisms in $\mathbb{R}^3$

Dual numbers were introduced by W.K. Clifford (1849-79) as a tool for his geometrical investigations. Then dual numbers have been used to study the motion of a straight line in $\mathbb{R}^3$, they even look like the most suitable tools for this goal. E.Study is first person that applied this [6] and since his time dual numbers have had an designated place in kinematics as a apparatus to make out pronlems being interested lines in space. Moreover E.Study used dual numbers and dual vectors in his research on the geometry of lines and kinematics [6]. The exercising of dual numbers to the lines of the Euclidean 3-space is performed by the principle of mobilisation which was facquired by E.Study. It warrants a complete generalization of the mathematical expression for the spherical point geometry to the spatial line geometry by means of dual number extension, i.e. replacing all ordinary quantities by the corresponding dual number quantities. Skreiner worked kinamatics of instantaneous spatial motion and he gave some theorems and results for the invariants of a closed ruled surface generated by an oriented line of a moving rigid body in $\mathbb{R}^3$ [21].

The angle of pitch, drall and lenght of pitch are called the integral invariants of a closed ruled surface. These topics are very important in the study of the geometry of lines from the perspectives of instant taneous space

kinematics and mechanism. Lost of authors have managed these integral invariants in their studies with regard to the generalization of some of the theorems of plane kinematics to spatialkinematics ([11], [27], [25], [3], [13], [14], [18], [9], [15]). The pitches and the angles of the pitches of the closed ruled surfaces corresponding to the one parameter dual unit spherical curves and oriented lines in $\mathbb{R}^3$ were calculated respectively by Hacsalihoglu [10], [11], [12] and Gursoy [7]. Some relations between the integral invariants of the closed ruled surfaces which correspond to the dual closed curves were calculated by Yapar, [28]. Köse gave the pitch and the angle of the pitch of the closed ruled surface in terms of the integral invariants of the dual spherical closed curve which corresponds to the closed ruled surface in 1997 [16]. Gürsoy and Küçük found some results with respect to geometric invariants of trajectory closed trajectory ruled surfaces, [8].

Definitions of the parallel ruled surface were presented by Wilhelm Blaschke [5]. The integral invariants of the parallel ruled surfaces in $\mathbb{R}^3$ corresponding to the unit dual spherical parallel curves were calculated by Şenyurt [20].

In this paper, we define the closed ruled surface which generated by a spacelike curve with timelike binormal. Then we calculate the pitch, the angle of pitch and drall of parallel ruled surface. More over we describe parallel ruled surface which produced by a spacelike curve with timelike binormal. We investigate the relations between the pitch length, the dual angle of pitch and drall of parallel ruled surface of a closed spacelike curve with timelike binormal in dual Lorentzian space. Finally we compute the pitch length, the dual angle of pitch and drall of the ruled surface which generated by a unit vectors $\vec{C}$ and $\vec{\bar{C}}$ in the instantaneous dual Darboux vectors $\vec{\psi}$ and $\vec{\bar{\psi}}$.

2 Preliminaries

In this chapter, we will give some notation and purport for our subject.

The set $D = \{\hat{\lambda} = \lambda + \varepsilon\lambda^* \mid \lambda, \lambda^* \in \mathbb{R}, \varepsilon^2 = 0\}$ is called dual numbers set [4]. D is an associative ring with the unit element 1.

On this set, product and addition operation are respectively,

$$(\lambda + \varepsilon\lambda^*) + (\beta + \varepsilon\beta^*) = (\lambda + \beta) + \varepsilon(\lambda^* + \beta^*)$$

and

$$(\lambda + \varepsilon\lambda^*)(\beta + \varepsilon\beta^*) = \lambda\beta + \varepsilon(\lambda\beta^* + \lambda^*\beta).$$

The elements of the set $ID^3 = \{\vec{A} = \vec{a} + \varepsilon\vec{a}^* \mid \vec{a}, \vec{a}^* \in IR^3\}$ are called dual vectors. On this set addition and scalar product operations are respectively

$$\oplus : ID^3 \times ID^3 \rightarrow ID^3$$

$$(\vec{A}, \vec{B}) \rightarrow \vec{A} \oplus \vec{B} = \vec{a} + \vec{b} + \varepsilon(\vec{a}^* + \vec{b}^*)$$

$$\odot : ID \times ID^3 \rightarrow ID^3$$

$$(\lambda, \vec{A}) \rightarrow \lambda \odot \vec{A} = (\lambda + \varepsilon\lambda^*) \odot (\vec{a} + \varepsilon\vec{a}^*) = \lambda\vec{a} + \varepsilon(\lambda\vec{a}^* + \lambda^*\vec{a})$$

The set $(ID^3, \oplus)$ is a module over the ring $(ID, +, \cdot)$.

The Lorentzian inner product of dual vectors $\vec{A}, \vec{B} \in ID^3$ is defined by

$$\langle \vec{A}, \vec{B} \rangle = \langle \vec{a}, \vec{b} \rangle + \varepsilon(\langle \vec{a}, \vec{b}^* \rangle + \langle \vec{a}^*, \vec{b} \rangle)$$

where $\langle \vec{a}, \vec{b} \rangle$ is following the Lorentzian inner product $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3) \in IR^3$

$$\langle \vec{a}, \vec{b} \rangle = -a_1b_1 + a_2b_2 + a_3b_3.$$

The set ID^3 equipped with the Lorentzian inner product $\langle \vec{A}, \vec{B} \rangle$ is called

3-dimensional dual Lorentzian space and denoted by $ID_1^3 = \{\vec{A} = \vec{a} + \varepsilon\vec{a}^* \mid \vec{a}, \vec{a}^* \in IR_1^3\}$ [24].

A dual vector $\vec{A} = \vec{a} + \varepsilon\vec{a}^* \in ID_1^3$ is called A dual space-like vector if $\langle \vec{A}, \vec{A} \rangle > 0$ or $\vec{A} = 0$, A dual time-like vector if $\langle \vec{A}, \vec{A} \rangle < 0$, A dual null(light-like) vector if $\langle \vec{A}, \vec{A} \rangle = 0$ for $\vec{A} \neq 0$. For $\vec{A} \neq 0$, the norm $\|\vec{A}\|$ of $\vec{A} = \vec{a} + \varepsilon\vec{a}^* \in ID^3$ is defined by

$$\|\vec{A}\| = \sqrt{|\langle \vec{A}, \vec{A} \rangle|} = \|\vec{a}\| + \varepsilon \frac{\langle \vec{a}, \vec{a}^* \rangle}{\|\vec{a}\|}, \quad \|\vec{a}\| \neq 0.$$

The dual Lorentzian cross-product of $\vec{A}, \vec{B} \in ID^3$ is defined as

$$\vec{A} \wedge \vec{B} = \vec{a} \wedge \vec{b} + \varepsilon (\vec{a} \wedge \vec{b}^* + \vec{a}^* \wedge \vec{b})$$

where $a \wedge b$ is the Lorentzian cross-product[24], this cross product given by

$$\vec{a} \wedge \vec{b} = (a_3b_2 - a_2b_3, a_1b_3 - a_3b_1, a_1b_2 - a_2b_1).$$

Theorem 2.1 *The oriented lines in are in one to one correspondence with the points of the dual unit sphere where ID-Modul , see [17].*

The Dual number $\Phi = \phi + \varepsilon\phi^*$ is called dual angle the unit dual vectors between $\vec{A}$ ve $\vec{B}$ and keep in mind that

$$\begin{aligned} \sinh(\phi + \varepsilon\phi^*) &= \sinh \phi + \varepsilon\phi^* \cosh \phi \\ \cosh(\phi + \varepsilon\phi^*) &= \cosh \phi + \varepsilon\phi^* \sinh \phi. \end{aligned}$$

3 On Some Characterizations of Ruled Surface of a Closed Spacelike Curve With Timelike Binormal in Dual Lorentzian Space

Let $U : I \rightarrow ID^3, t \rightarrow \vec{U}(t) = \vec{U}_1(t), \|\vec{U}(t)\| = 1$ be a differentiable spacelike curve with timelike binormal in the dual unit sphere. Denote by U the closed ruled surface of a spacelike curve with timelike binormal generated by this curve. Let $\{\vec{U}_1, \vec{U}_2, \vec{U}_3\}$ be the dual Frenet frame of the curve $\vec{U} = \vec{U}_1$ with

$$\vec{U}_1 = \vec{U}(t) \quad , \quad \vec{U}_2 = \vec{U}'(t) / \|\vec{U}(t)\| \quad , \quad \vec{U}_3 = \vec{U}_1 \wedge \vec{U}_2$$

Now, take $\vec{U}(t)$ as a closed spacelike curve with timelike binormal curve with curvature $\kappa = k_1 + \varepsilon k_1^*$ and torsion $\tau = k_2 + \varepsilon k_2^*$. Recall that in the

Frenet frame associated to the curve U , $\vec{U}_1$ and $\vec{U}_2$ are spacelike vectors, $\vec{U}_3$ is timelike vector and we have

$$\vec{U}_1 \wedge \vec{U}_2 = \vec{U}_3, \quad \vec{U}_2 \wedge \vec{U}_3 = -\vec{U}_1, \quad \vec{U}_3 \wedge \vec{U}_1 = -\vec{U}_2.$$

Under these conditions, the Frenet formulas are given by [26]

$$\vec{U}_1' = \kappa \vec{U}_2, \quad \vec{U}_2' = -\kappa \vec{U}_1 + \tau \vec{U}_3, \quad \vec{U}_3' = \tau \vec{U}_2.$$

The real and dual parts of $\vec{U}_1', \vec{U}_2', \vec{U}_3'$ are

$$\left\{ \begin{array}{l} \vec{u}_1' = k_1 \vec{u}_2, \\ \vec{u}_2' = -k_1 \vec{u}_1 + k_2 \vec{u}_3, \\ \vec{u}_3' = k_2 \vec{u}_2, \\ \vec{u}_1'^* = k_1^* \vec{u}_2 + k_1 \vec{u}_2^*, \\ \vec{u}_2'^* = -k_1^* + k_2^* \vec{u}_3 - k_1 \vec{u}_1^* + k_2 \vec{u}_3^*, \\ \vec{u}_3'^* = k_2^* \vec{u}_2 + k_2 \vec{u}_2^*. \end{array} \right.$$

The Frenet instantaneous rotation vector (also called instantaneous Darboux vector) for the spacelike curve with timelike binormal $\vec{U}(t)$ curve is given by [23]

$$\vec{\Psi} = \tau \vec{U}_1 - \kappa \vec{U}_3, \quad (1)$$

Under these condition the so called Steiner vector of the motion is which can be written as

$$\vec{D} = \vec{U}_1 \oint \tau dt - \vec{U}_3 \oint \kappa dt.$$

The real and dual parts of $\vec{D}$ are

$$\left\{ \begin{array}{l} \vec{d} = \vec{u}_1 \oint k_2 dt - \vec{u}_3 \oint k_1 dt, \\ \vec{d}^* = \vec{u}_1^* \oint k_2 dt + \vec{u}_1 \oint k_2^* dt - \vec{u}_3^* \oint k_1 dt - \vec{u}_3 \oint k_1^* dt \end{array} \right.$$

In the following theorems, we find the pitch, the dual angle of pitch and drall of the respective closed ruled surfaces, $\mathbf{U}_1, \mathbf{U}_2, \mathbf{U}_3$.

We find the pitch, the dual angle of pitch and drall of the closed ruled surfaces which generated by a spacelike curve with timelike binormal in the following theorem.

Theorem 3.1 *The pitch, the dual angle of pitch and drall of the closed ruled surface U_2 are given by*

$$1) L_{u_1} = \oint k_2^* dt \quad 2) \Lambda_{U_1} = 0 \quad 3) P_{U_1} = \frac{k_1^*}{k_1}$$

Proof. Starightforward

Theorem 3.2 *The pitch, the dual angle of pitch and drall of the closed ruled surface U_1 are given by*

$$1) L_{u_2} = 0 \quad 2) \Lambda_{U_2} = 0 \quad 3) P_{U_2} = \frac{k_1 k_1^* - k_2 k_2^*}{k_1^2 - k_2^2}$$

Proof. Starightforward

Let $\Omega(t) = \omega(t) + \varepsilon \omega^*(t)$ be Lorentzian timelike angle of between the instantaneous dual Pfaffion vector $\vec{\Psi}$ and the vector $\vec{U}_3$

a) If the instantaneous dual Pfaffion vector $\vec{\Psi}$ is spacelike ($|\tau| > |\kappa|$)

$$\kappa = \|\vec{\Psi}\| \sinh \Omega \quad , \quad \tau = \|\vec{\Psi}\| \cosh \Omega$$

On the way $\vec{C} = \vec{\mathcal{C}} + \varepsilon \vec{\mathcal{C}}^*$, unit vector about the vector $\vec{\Psi}$ direction is

$$\vec{C} = \cosh \Omega \vec{U}_1 - \sinh \Omega \vec{U}_3 \quad (2)$$

If the equation (2) is separated into the dual and real part, we can obtain

$$\begin{cases} \vec{\mathcal{C}} = \cosh \omega \vec{u}_1 - \sinh \omega \vec{u}_3 \\ \vec{\mathcal{C}}^* = \cosh \omega \vec{u}_1^* - \sinh \omega \vec{u}_3^* + \omega^* \sinh \omega \vec{u}_1 - \omega^* \cosh \omega \vec{u}_3 \end{cases}$$

Theorem 3.3 *The pitch, the dual angle of pitch and drall of the closed ruled surface $\vec{C}$ are given by*

$$1) L_C = \cosh \omega L_{u_1} - \sinh \omega L_{u_3} - \omega^* (\sinh \omega \lambda_{u_1} - \cosh \omega \lambda_{u_3})$$

$$2) \Lambda_C = \cosh \Omega \Lambda_{U_1} - \sinh \Omega \Lambda_{U_3}$$

$$3) P_C = \frac{\omega' \omega^* + (k_1 \sinh \omega - k_2 \cosh \omega) [(k_1^* - k_2 \omega^*) \sinh \omega + (k_1 \omega^* - k_2^*) \cosh \omega]}{(k_1 \sinh \omega - k_2 \cosh \omega)^2 + \omega'^2}$$

where λ_{u_1} and λ_{u_3} real part of Λ_{U_1} and Λ_{U_3} , respectively.

Proof. Starightforward

b) If the instantaneous dual Pfaffion vector $\vec{\Psi}$ is timelike ($|\tau| < |\kappa|$)

$$\kappa = \|\vec{\Psi}\| \cosh \Omega, \quad \tau = \|\vec{\Psi}\| \sinh \Omega$$

On the way $\vec{C} = \vec{c} + \varepsilon \vec{c}^*$, unit vector about the vector $\vec{\Psi}$ direction is

$$\vec{C} = \sinh \Omega \vec{U}_1 - \cosh \Omega \vec{U}_3 \quad (3)$$

If the equation (3) is separated into the dual and real part, we can obtain

$$\begin{cases} \vec{c} = \sinh \omega \vec{u}_1 - \cosh \omega \vec{u}_3 \\ \vec{c}^* = \sinh \omega \vec{u}_1^* - \cosh \omega \vec{u}_3^* + \omega^* \cosh \omega \vec{u}_1 - \omega^* \sinh \omega \vec{u}_3 \end{cases}$$

Theorem 3.4 *The pitch, the dual angle of pitch and drall of the closed ruled surface $\vec{C}$ are given by*

$$1) L_C = \sinh \omega L_{u_1} - \cosh \omega L_{u_3} - \omega^* (\cosh \omega \lambda_{u_1} - \sinh \omega \lambda_{u_3})$$

$$2) \Lambda_C = -\cosh \Omega \Lambda_{U_1} + \sinh \Omega \Lambda_{U_3}$$

$$3) P_C = \frac{\omega^{*'} + (k_1 \sinh \omega - k_2 \cosh \omega) [(k_1^* - k_2 \omega^*) \sinh \omega + (k_1 \omega^* - k_2^*) \cosh \omega]}{(k_1 \sinh \omega - k_2 \cosh \omega)^2 + \omega'^2}$$

where λ_{u_1} and λ_{u_3} real part of Λ_{U_1} and Λ_{U_3} , respectively.

Proof. Starightforward

Definition 3.1 *The closed ruled surface U corresponding to the dual space-like curve with timelike binormal curve $\overrightarrow{U(t)}$ which makes a fixed dual angle $\Phi = \varphi + \varepsilon \varphi^*$ with $\overrightarrow{U(t)}$ determines*

$$\vec{V} = \cosh \Phi \vec{U}_1 + \sinh \Phi \vec{U}_3 \quad (4)$$

The surface $\mathbf{V}$ corresponding to the dual spacelike vector $\vec{V}$ is called the parallel ruled surface U in the dual Lorentzian space D_1^3 .

Let be $\vec{V}_1 = \vec{V}$. Differentiating of the vector $\vec{V}_1$ with respect to the parameter t and using the Frenet formulas we get

$$\vec{V}_1' = (\kappa \cosh \Phi + \tau \sinh \Phi) \vec{U}_2 \quad (5)$$

and the norm of the vector denoted by is

$$P = \kappa \cosh \Phi + \tau \sinh \Phi \quad (6)$$

Then, we get

$$\vec{V}_2 = \vec{U}_2$$

For the vector $\vec{V}_3$, we have

$$\vec{V}_3 = \sinh \Phi \vec{U}_1 + \cosh \Phi \vec{U}_3$$

The real and dual parts of $\vec{U}_1, \vec{U}_2, \vec{U}_3$ are

$$\left\{ \begin{array}{l} \vec{u}_1 = \cosh \varphi \vec{v}_1 - \sinh \varphi \vec{v}_3 \\ \vec{u}_2 = \vec{v}_2 \\ \vec{u}_3 = -\sinh \varphi \vec{v}_1 + \cosh \varphi \vec{v}_3 \\ \vec{u}_1^* = \cosh \varphi \vec{v}_1^* - \sinh \varphi \vec{v}_3^* + \varphi^* (\sinh \varphi \vec{v}_1 - \cosh \varphi \vec{v}_3) \\ \vec{u}_2^* = \vec{v}_2^* \\ \vec{u}_3^* = -\sinh \varphi \vec{v}_1^* + \cosh \varphi \vec{v}_3^* - \varphi^* (\cosh \varphi \vec{v}_1 - \sinh \varphi \vec{v}_3) \end{array} \right.$$

Let $P = p + \varepsilon p^*$ be the curvature and $Q = q + \varepsilon q^*$ the torsion of curve $\vec{V}(t)$. Then, the following relating holds between the vectors $\vec{V}_1, \vec{V}_2, \vec{V}_3$ and the $\vec{V}_1', \vec{V}_2', \vec{V}_3'$ [24]

$$\left\{ \begin{array}{l} \vec{V}_1' = P \vec{V}_2', \quad \vec{V}_1' = -P \vec{V}_1 + Q \vec{V}_3, \quad \vec{V}_3' = Q \vec{V}_2 \\ P = \sqrt{\langle \vec{V}_1', \vec{V}_1' \rangle}, \quad Q = \frac{\det(\vec{V}_1, \vec{V}_1', \vec{V}_1'')}{\langle \vec{V}_1, \vec{V}_1' \rangle}. \end{array} \right. \quad (7)$$

If the equation (7) is separated into the real and dual parts, we can write

$$\begin{cases} \vec{v}_1' = p \vec{v}_2, & \vec{v}_2' = -p \vec{v}_1 + q \vec{v}_3, & \vec{v}_3' = q \vec{v}_2 \\ \vec{v}_1'^* = p \vec{v}_2^* + p^* \vec{v}_2, \\ \vec{v}_2'^* = -p \vec{v}_1^* - p^* \vec{v}_1 + q^* \vec{v}_3, \\ \vec{v}_3'^* = q \vec{v}_2^* + q^* \vec{v}_2 \end{cases}$$

Now, we are ready calculate the value of Q relative to κ and τ . Deriving the equation (5) with respect to the curve parameter t we get

$$\begin{aligned} \vec{v}_1'' = & (-\kappa^2 \cosh \Phi - \kappa \tau \sinh \Phi) \vec{U}_1 + \\ & + (\kappa \cosh \Phi + \tau \sinh \Phi)' \vec{U}_2 + (\kappa \tau \cosh \Phi + \tau^2 \sinh \Phi) \vec{U}_3 \end{aligned} \quad (8)$$

Substituting by the equations (4) , (5) and (8) into the equation (7) , we get

$$Q = \kappa \sinh \Phi + \tau \cosh \Phi \quad (9)$$

The equations (6) and (9) are separated into the dual and real parts, we have

$$\begin{cases} p = k_1 \cosh \varphi + k_2 \sinh \varphi \\ p^* = k_1^* \cosh \varphi + k_2^* \sinh \varphi + \varphi^* (k_1 \sinh \varphi + k_2 \cosh \varphi) \\ q = k_1 \sinh \varphi + k_2 \cosh \varphi \\ q^* = k_1^* \sinh \varphi + k_2^* \cosh \varphi + \varphi^* (k_1 \cosh \varphi + k_2 \sinh \varphi) \end{cases}$$

In its dual unit spherical motion the dual orthonormal system $\{\vec{V}_1, \vec{V}_2, \vec{V}_3\}$ at any t makes a dual rotation motion around the instantaneous dual Darboux vector. This vector is determined the following equation[23]

$$\vec{\Psi} = Q \vec{V}_1 - P \vec{V}_3, [13] \quad (10)$$

Fort he Steiner vector of the motion, we can write

$$\vec{D} = \oint \vec{\Psi}$$

Then, from (1) we have

$$\vec{\Psi} = Q\vec{V}_1 - P\vec{V}_3.$$

So, in this case, taking into account the last equation and (10), we have

$$\vec{\Psi} = \vec{\Psi}$$

Since $\vec{D} = \oint \vec{\Psi}$ and $\vec{D} = \oint \vec{\Psi}$ we can obtain $\vec{D} = \vec{D}$. Then, for the dual Steiner vector of taht motion, we have

$$\vec{D} = \vec{V}_1 \oint Q dt - \vec{V}_3 \oint P dt$$

and its real and dual parts are

$$\begin{cases} \vec{d} = \vec{v}_1 \oint q dt - \vec{v}_3 \oint p dt, \\ \vec{d}^* = \vec{v}_1 \oint q^* dt + \vec{v}_1^* \oint q dt + \vec{v}_1^* \oint q dt - \vec{v}_3^* \oint p dt. \end{cases}$$

Theorem 3.5 *Let V_1 be the parallel surface of the surface U_1 . The pitch, drall and the dual of the pitch of the ruled surface V_1 are*

$$1-)L_{V_1} = \oint q^* dt \quad 2-)\Lambda_{V_1} = - \oint Q dt \quad 3-)P_{V_1} = \frac{p^*}{p}$$

Proof. Starightforward

Conclusion 3.1 *Let V_1 be the parallel surface of the surface U_1 . The pitch and the dual of the pitch of the ruled surface V_1 related to the inavariants of the surface U_1 are given by*

$$1)L_{V_1} = \cosh \varphi L_{u_1} + \sinh \varphi L_{u_3} - \varphi^* (\sinh \varphi \lambda_{u_1} + \cosh \varphi \lambda_{u_3})$$

$$2)\Lambda_{V_1} = \cosh \phi \Lambda_{U_1} + \sinh \phi \Lambda_{U_3}$$

$$3)P_{V_1} = \frac{k_1^* \cosh \varphi + k_2^* \sinh \varphi}{k_1 \cosh \varphi + k_2 \sinh \varphi} + \varphi^* \frac{k_1 \sinh \varphi + k_2 \cosh \varphi}{k_1 \cosh \varphi + k_2 \sinh \varphi}.$$

where λ_{u_1} and λ_{u_3} real part of Λ_{U_1} and Λ_{U_3} , respectively.

Theorem 3.6 *Let V_1 be the parallel surface of the surface U_1 . The pitch, drall and the dual of the pitch of the ruled surface V_2 are*

$$1) L_{V_2} = 0 \quad 2) \Lambda_{V_2} = 0 \quad 3) P_{V_2} = \frac{pp^* - qq^*}{p^2 - q^2}$$

Proof. Straightforward

Conclusion 3.2 *Let V_1 be the parallel surface of the surface U_1 . The pitch and the dual of the pitch of the ruled surface V_1 related to the invariants of the surface U_1 are given by*

$$\begin{aligned} 1) L_{V_2} &= L_{u_1} \\ 2) \Lambda_{V_1} &= \Lambda_{U_2} \\ 3) P_{V_1} &= \frac{k_1 k_1^* - k_2 k_2^*}{k_1^2 - k_2^2}. \end{aligned}$$

Theorem 3.7 *Let V_1 be the parallel surface of the surface U_1 . The pitch, drall and the dual of the pitch of the ruled surface V_3 are*

$$1) L_{V_3} = \oint p^* dt \quad 2) \Lambda_{V_3} = - \oint P dt \quad 3) P_{V_3} = \frac{q^*}{q}$$

Proof. Straightforward

Conclusion 3.3 *Let V_1 be the parallel surface of the surface U_1 . The pitch and the dual of the pitch of the ruled surface V_3 related to the invariants of the surface U_1 are given by*

$$\begin{aligned} 1) L_{V_3} &= \sinh \phi L_{u_1} + \cosh \phi L_{u_3} - \phi^* (\cosh \phi \lambda_{u_1} + \sinh \phi \lambda_{u_3}) \\ 2) \Lambda_{V_3} &= \sinh \Phi \Lambda_{U_1} + \cosh \Phi \Lambda_{U_3} \end{aligned}$$

$$3)P_{V_3} = \frac{k_1^* \sinh \varphi + k_2^* \cosh \varphi}{k_1 \sinh \varphi + k_2 \cosh \varphi} + \varphi^* \left(\frac{k_1 \cosh \varphi + k_2 \sinh \varphi}{k_1 \sinh \varphi + k_2 \cosh \varphi} \right)$$

where λ_{u_1} and λ_{u_3} real part of Λ_{U_1} and Λ_{U_3} , respectively.

Let $\Theta(t) = \theta(t) + \varepsilon \theta^*(t)$ be Lorentzian timelike angle of between the instantaneous dual Darboux vector $\vec{\Psi}$ and the vector $\vec{V}_3$.

a) If the instantaneous dual Darboux vector $\vec{\Psi}$ is spacelike ($|Q| > |P|$)

$$P = \left\| \vec{\Psi} \right\| \sinh \Theta, \quad Q = \left\| \vec{\Psi} \right\| \cosh \Theta.$$

The way unit vector $\vec{C} = \vec{c} + \varepsilon \vec{c}^*$ in the $\vec{\Psi}$ direction is

$$\vec{C} = \cosh \Theta \vec{V}_1 - \sinh \Theta \vec{V}_3$$

We can easily get

$$\vec{C} = \cosh (\Theta - \Phi) \vec{U}_1 - \sinh (\Theta - \Phi) \vec{U}_3$$

The real and dual parts of $\vec{C}$ are

$$\begin{aligned} \vec{c} &= \cosh \theta \vec{v}_1 - \sinh \theta \vec{v}_3 \\ \vec{c}^* &= \cosh \theta \vec{v}_1^* - \sinh \theta \vec{v}_3^* + \theta^* \sinh \theta \vec{v}_1 - \theta^* \cosh \theta \vec{v}_3 \end{aligned}$$

Theorem 3.8 The pitch, the dual angle of pitch and drall of the closed ruled surface $\vec{C}$ are given by

$$\begin{aligned} 1) L_{\vec{C}} &= \cosh (\theta - \phi) L_{U_1} - \sinh (\theta - \phi) L_{U_3} + (\phi^* - \theta^*) (\sinh (\theta - \phi) \lambda_{U_1}) \\ &\quad - \cosh (\theta - \phi) \lambda_{U_3} \end{aligned}$$

$$2) \Lambda_{\vec{C}} = \cosh \Omega \Lambda_{V_1} - \sinh \Omega \Lambda_{V_3}$$

$$3) P_{\vec{C}} = \frac{\omega'^* + (k_1 \sinh \omega - k_2 \cosh \omega) [(k_1^* - k_2 \omega^*) \sinh \omega + (k_1 \omega^* - k_2^*) \cosh \omega]}{(k_1 \sinh \omega - k_2 \cosh \omega)^2 - \omega'^2}$$

where λ_{u_1} and λ_{u_3} real part of Λ_{U_1} and Λ_{U_3} , respectively.

Proof. Starightforward

b) If the instantaneous dual Dual vector $\vec{\Psi}$ is timelike ($|Q| < |P|$)

$$P = \left\| \vec{\Psi} \right\| \cosh \Theta \quad , \quad Q = \left\| \vec{\Psi} \right\| \sinh \Theta$$

The way unit vector $\vec{C} = \vec{c} + \varepsilon \vec{c}^*$ in the $\vec{\Psi}$ direction is

$$\vec{C} = \sinh \Theta \vec{V}_1 - \cosh \Theta \vec{V}_3$$

We can easily get

$$\vec{C} = \sinh (\Theta - \Phi) \vec{U}_1 + \cosh (\Theta - \Phi) \vec{U}_3$$

The real and dual parts of $\vec{C}$ are

$$\begin{cases} \vec{c} = \sinh \theta \vec{v}_1 - \cosh \theta \vec{v}_3 \\ \vec{c}^* = \sinh \theta \vec{v}_1^* - \cosh \theta \vec{v}_3^* + \theta^* \cosh \theta \vec{v}_1 - \theta^* \sinh \theta \vec{v}_3 \end{cases}$$

Theorem 3.9 *The pitch, the dual angle of pitch and drall of the closed ruled surface $\vec{C}$ a are given by*

$$\begin{aligned} 1) L_{\vec{C}} &= \sinh (\theta - \varphi) L_{U_1} - \cosh (\theta - \varphi) L_{U_3} + (\varphi^* - \theta^*) (\cosh (\theta - \varphi) \lambda_{U_1}) \\ &\quad - \sinh (\theta - \varphi) \lambda_{U_3} \end{aligned}$$

$$2) \Lambda_{\vec{C}} = \sinh (\Theta - \Phi) \Lambda_{U_1} - \cosh (\Theta - \Phi) \Lambda_{U_3}$$

$$3) P_{\vec{C}} = \frac{\theta'^* + (p \sinh \theta - q \cosh \theta) [(p \theta^* - q^*) \cosh \theta + (p^* - q \theta^*) \sinh \theta]}{(p \sinh \theta - q \cosh \theta)^2 + \theta'^2}$$

where λ_{u_1} and λ_{u_3} real part of Λ_{U_1} and Λ_{U_3} , respectively.

Proof. Starightforward

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**INDICATION OF SUBSISTENCE OF MAGNETIC CHARGE
FROM 70 MeV MASS QUANTA**

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Abstract

In this contribution the authentication of the comprehensive constituent quark model exhibiting the 70 MeV mass quanta involved in hadron spectroscopy has been outlined with respect to the existence of magnetic charge. The light meson spectra available from Particle Data Group-2008 listings has being viewed to display Zeeman splitting upon scrutiny of the non-Regge spacing with $J \sim M$. Evidence of L-S coupling is derived for the meson states satisfying the Lande's interval rule and subsequently the involvement of magnetic interactions is displayed leading to subsistence of magnetic monopole.

Keywords: Constituent quark model, magnetic monopole, hadron masses, Zeeman splitting, non-Regge plots.

1. Introduction

Mac Gregor [1,2,3] developed a comprehensive constituent quark (CQ) model of elementary particles and noted regularities in the hadron spectrum with 70, 140 and 210 MeV energy separations. Later a connection was discovered between CQ masses and Nambu's empirical mass formula which is based upon the existence of magnetic charge, $m_n = \frac{n}{2} 137m_e$, n being a positive integer and m_e being the mass of the electron [4]. Thus, the 70MeV mass quantum [5] of Mac Gregor has its origin in the existence of Dirac magnetic charge. The experimental data from the Particle Data Group 2010 listings [6] support the existence of a 70 MeV mass quanta. This mass quanta can also be easily derived from electron mass and fine structure constant as $\frac{m_e}{\alpha} = 70MeV$. Moreover, it has also been shown [7] that Frosch's derivation of the empirical mass formula $m \sim N(m_e)$, where N is an integer, has as its basis the existence of Dirac unit of magnetic charge $g = \left(\frac{137}{2}\right)e = 68.5e$. The concept of quantized hadronic masses has also been given [8], taking all hadron masses in the multiple of 70 MeV which leads to linearly rising Regge trajectories (J versus M^2 plots).

The linear reliance of J and M^2 is a direct consequence of the linear potential between quarks at large separation and reveals the fact that

confinement aspect of QCD is directly related to the straight line equation of Regge trajectories. Eventually, the results [8] supported the 70 MeV mass quanta as the ultimate hadronic building block. Thereby, evidence suggests that 70 MeV mass quanta should have its origin traced from the mass of classical Dirac magnetic monopole and the contemporaneous work focusses on this particular aspect.

In the present work, efforts are made to derive the existence of magnetic charge from the study of light meson spectroscopy using experimental data available from Particle Data Group listings 2008 [9]. The method devised to find out the evidence of Dirac's mass magnetic monopole traces path from the detection of magnetic charge inside hadrons through observing Zeeman splitting in non Regge –spacing plots.

In light meson spectra, Russell-Saunders coupling scheme has been applied which is shown to satisfy the Lande's interval rule. Zeeman splitting in low mass mesons is observed which lifts the degeneracy of P-states and generate distinct particles and the obtained patterns and amount of splitting are a signature that magnetic field is present. It indicates that strong magnetic fields of composite mesons are responsible for the mass splitting and thus provides evidence for magnetic charge in hadronic structure.

2. Existence of magnetic charge in Meson states

Constant efforts are made by physicists to prove the existence of magnetic monopole predicted by Dirac. There have been proposals by Schwinger and others [10-18] that quarks may consist of both electric and magnetic charge. It was suggested [19] that magnetic charge of spin $J=0$ may exist internal to the quarks and generate magnetic fields among the meson states. Evidence of magnetic

charge in scattering experiments was also given [20] to account for residual strong interaction effects. There is evidence [21] that a low mass magnetic monopole of Dirac charge $g = \left(\frac{137}{2}\right)e$ may be Zeeman-splitting meson states. A quantum mechanical derivation of the Zeeman effect for an electric charge – magnetic monopole system has also been given [22]. In hadron spectroscopy, Zeeman splitting is possible in the presence of magnetic charge. All these findings further strengthens the idea of the possible existence of magnetic charge in hadronic structure. In the present work we have shown the presence of Zeeman splitting among the meson states by plotting non-Regge spacing with $J \sim M$ for low mass mesonic data available from Particle Data Group listings 2008 [9]. Here we have applied the Russell-Saunders coupling scheme [23] to the quarks in light meson spectroscopy. Keeping in mind the fact that mesons, composite of quark-antiquark pairs are extensively studied as quarkonium and have been compared to positronium like bound states, the physics of the atomic scale has been applied to particle scale [19]. Therefore splitting is expected in P-states of meson system, with $3P_1$ states splitting into 3 separate levels, and with $3P_2$ states splitting into 5 separate levels in the presence of magnetic field. We have examined the experimental evidence of P-level splitting in the light meson systems. In figures (1,2,3) there are indications of particle states which fit L-S coupling scheme. Solid lines indicate experimental masses and those indicated by dashed lines are predicted to exist.

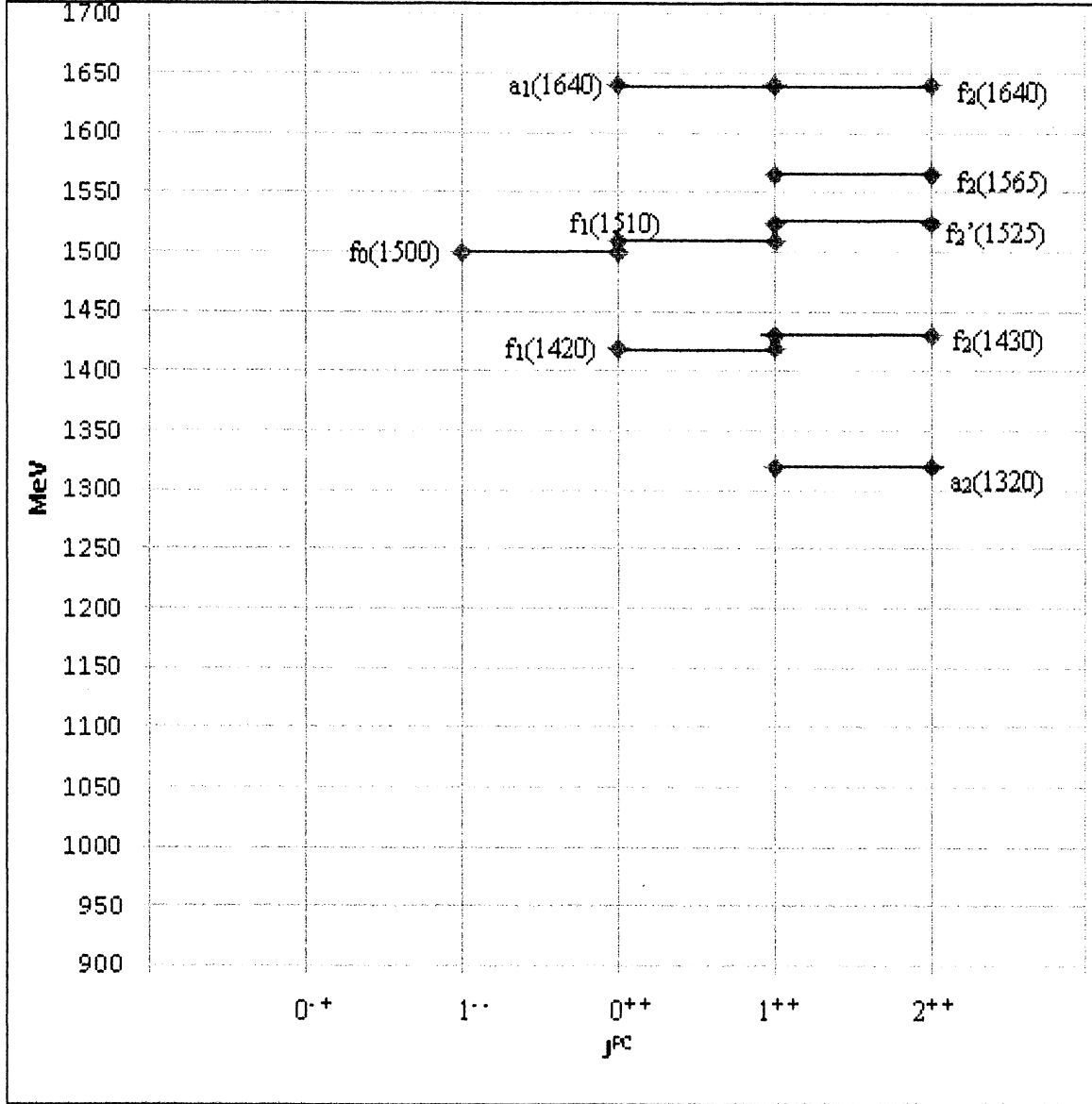


Fig. 1: The set of low meson masses connected with $f_0(1500)$ state. Experimental meson masses are indicated with solid lines and satisfy Lande's interval rule.

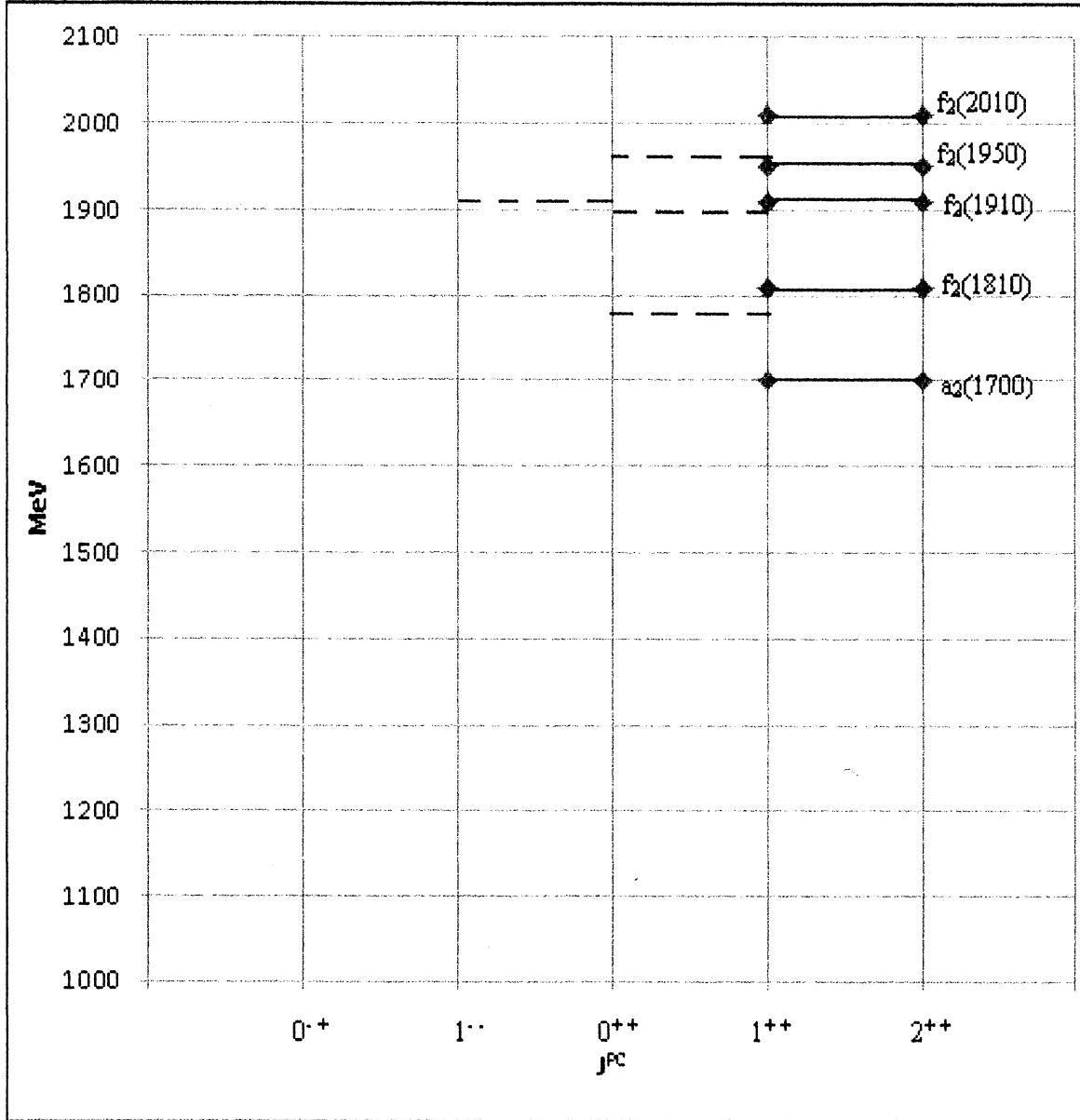


Fig. 2: The set of low meson masses without connection to a singlet state but satisfy the Lande's interval rule. Experimental meson masses are indicated with solid lines. Those indicated by dashed-dot lines are predicted to exist.

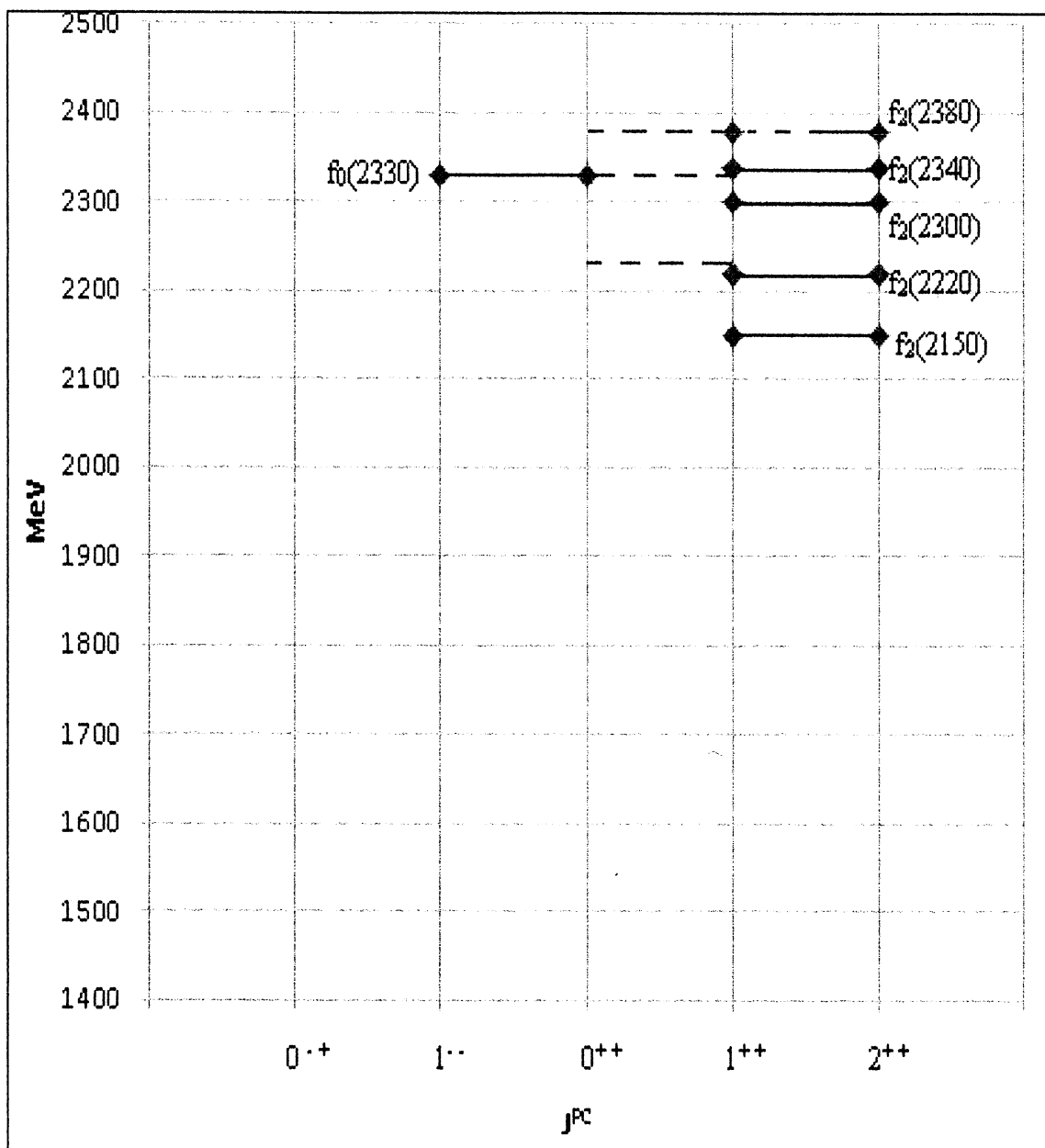


Fig. 3: The set of low meson masses connected with $f_0(2330)$ state.

Solid lines represent experimental masses. Those indicated by dashed-dot lines are predicted to exist and also satisfy the Lande's interval rule.

Thus the evidence of L-S coupling is derived from the light meson spectra. If these spectra are shown to satisfy Lande interval rule, which is widely used in atomic molecular and nuclear physics it gives support to the L-S coupling. For the P-state ($J=2$) the Lande interval rule predicts a mass splitting or ratio of 2.0 for the states within the same multiplet. We have calculated Lande ratio for these mass splitting (Table 1).

Table 1- Calculations of the mass splitting for P-states ($J=2$) associated with mesons in Fig 1, Fig 2 and Fig 3

Figure	Calculation of Lande Interval	Lande Ratio
Fig 1	Upper states: $\frac{[f_2(1640) - f_2'(1525)]}{[f_2'(1565) - f_2'(1525)]}$	2.8
	Lower states: $\frac{[f_2'(1525) - a_2(1320)]}{[f_2'(1525) - f_2(1430)]}$	2.1
Fig 2	Upper states: $\frac{[f_2(2010) - f_2(1910)]}{[f_2(1950) - f_2(1910)]}$	2.5
	Lower states: $\frac{[f_2(1910) - a_2(1700)]}{[f_2(1910) - f_2(1810)]}$	2.1
Fig 3	Upper states: $\frac{[f_2(2380) - f_2(2300)]}{[f_2(2340) - f_2(2300)]}$	2.0
	Lower states: $\frac{[f_2(2300) - f_2(2150)]}{[f_2(2300) - f_2(2220)]}$	1.9

Though the mass splitting can be due to various effects such as Zeeman effect, Paschen back effect and also Stark effect but here

we are interested in the effect which is due to the magnetic field and as mentioned earlier there are several evidences to believe that Zeeman effect is responsible for splitting of states, for lifting the degeneracy of P-states and for generating distinct particles.

The expression for energy in the Zeeman or Paschen Back splitting of the meson states is given by the relation

$$\Delta E = g m_j \mu B$$

Where g is Lande factor, μ is the magnetic dipole moment of the quark and B is the magnetic field which would have been generated because of the presence of magnetic charge inside the quarks. The individual quark's magnetic moment would then interact with the field B. It is speculated [24] that typical magnetic interactions are of 50 MeV depending upon the rms radius of the charge distribution for the mesons ($r \sim 0.6$ fm).

Furthermore, in a very interesting paper Akers [25] has proposed that the observed symmetry of meson states about 2397 MeV, is due to Zeeman splitting from quark's magnetic dipole moment interacting with monopole's magnetic field. This symmetry was obtained by non-Regge spacing [J~M] plot for light mass mesons and charmonium states. Here 2397 MeV is the first magnetic monopole mass i.e. Dirac mass and it was estimated on the basis of magnetic self-interaction [26,27]

$$M = \frac{g^2}{e^2} m_e$$

and according to Dirac quantization condition [28]

$$g_e = \frac{1}{2} n \hbar c, n=0, \pm 1, \dots$$

g is the magnetic charge of monopole, e is the electric charge and m_e is the electron mass.

While grand unification theory (GUT) predict massive 10^{16} GeV monopole [29], some physicists [30,31] studied low mass magnetic

monopoles. Though the existence of a 2397 MeV monopole has been reconciled with GUT [25].

Conclusions

The postulation of existence of Dirac's monopole is revealed indirectly through observation of possible Zeeman splitting in hadronic structure. Low mass meson states effectively display this splitting which is portrayed by non-Regge spacing plots with $J \sim M$ derived from Particle Data Group listings 2008. Thereby, experimental evidence for Russell- Saunders splitting is presented with testimonial to light meson spectroscopy, wherein tabulated data is found to satisfy Lande's interval rule. Furthermore, the famous 70 MeV mass quanta in itself serves as a keynote element for this entire postulation and thus projects out yet another unknown realm of particle interiors.

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**NEW MATHEMATICAL METHODS FOR GENERAL
DESCRIPTION OF A DIRAC PARTICLE IN GRAVITATIONAL FIELDS**

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Abstract

New mathematical methods developed in framework of the relativistic quantum mechanics are used for description of a Dirac particle in arbitrarily strong gravitational fields. The contemporary methods of the Foldy-Wouthuysen transformation allow to derive Foldy-Wouthuysen Hamiltonians for a relativistic particle in arbitrarily strong gravitational fields. The general Hermitian Dirac Hamiltonian is derived and transformed to the Foldy-Wouthuysen representation for the spatially isotropic metric. Quantum-mechanical equations of motion are obtained and their classical limit is found. The comparison of the quantum mechanical and classical equations shows their complete agreement. The Mathisson-Papapetrou equations are discussed. The general noninertial (accelerated and rotating) frame is considered. Evolution of helicity and the Lense-Thirring effect are calculated for relativistic particles. Squaring the covariant Dirac equation explicitly shows a similarity of the interactions of electromagnetic and gravitational fields with a Dirac particle.

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1 Introduction

The quantum and semiclassical description of a Dirac particles in gravitational fields is an important part of the contemporary theoretical physics. The main attention was usually paid to the dynamics of fermions in weak gravitational fields when the components of the spacetime metric tensor do not significantly deviate from the flat Minkowski metric. The analysis of strong gravitational fields is of interest for the investigation of processes near black holes and massive compact astrophysical objects. They may be also applied to describe the spaces of variable geometry and dimensionality discussed recently [1, 2, 3]. In this paper, we use the contemporary methods [4, 5] of transformation of initial quantum-mechanical equations to the Foldy-Wouthuysen (FW) representation (FW transformation) to describe a relativistic particle in arbitrarily strong gravitational fields. Specifically, we derive the general equation for the angular velocity of spin precession and compare this quantum-mechanical result with the classical one.

The paper is organized as follows. In Sec. 2, we expound the contemporary methods [4, 5] of the FW transformation. In Sec. 3, we present the derivation of the general Hermitian Dirac Hamiltonian for a charged particle interacting with the electromagnetic fields in an arbitrary curved spacetime. In this general framework, we further specialize to the dynamics of the spin and momentum of the particle in a spatially isotropic metric. Sec. 4 presents the FW Hamiltonian and operator equations of motion. The equations describing the dynamics of a classical spin are obtained in Sec. 5 and we compare them with the corresponding quantum equations, testing the equivalence principle. The evolution of the helicity in the strong fields is discussed in Sec. 6. The validity of the equivalence principle as well as the similarity of the spin interactions with the gravitational and with the electromagnetic fields is also manifested in the analysis of squared covariant Dirac equation performed in Sec. 7. The results obtained are summarized in Sec. 8. We denote world indices by Latin letters $i, j, k, \dots = 0, 1, 2, 3$ and reserve first Greek letters for tetrad indices, $\alpha, \beta, \dots = 0, 1, 2, 3$. Spatial indices are denoted by Latin letters from the beginning of the alphabet, $a, b, c, \dots = 1, 2, 3$. The temporal and spatial tetrad indices are distinguished by hats.

2 New mathematical methods in relativistic quantum mechanics

New mathematical methods developed in this century significantly change the form of the relativistic quantum mechanics and extraordinarily simplify the semiclassical transformation and finding the classical limit. These methods use the transformation of initial quantum-mechanical equations to the FW representation (FW transformation). Unlike the previously elaborated methods of this transformation (see the review [6] and references therein), the new methods enable obtaining the FW Hamiltonian and equations of motion in the FW representation for a particle in arbitrarily strong external fields.

The FW representation discovered by Foldy and Wouthuysen in 1950 [7] holds a special place in quantum mechanics thank to its unique properties. In this representation, quantum-mechanical operators for relativistic particles in external fields have the same form as in the nonrelativistic quantum theory. In particular, the position operator [8] and the momentum one are equal to $\mathbf{r}$ and $\mathbf{p} = -i\hbar\nabla$, respectively. The polarization operator for spin-1/2 particles is expressed by the Dirac matrix $\boldsymbol{\Pi}$. In other representations, these operators are expressed by much more cumbersome formulas (see Refs. [4, 7]). The relations between the operators in the FW representation are analogous to the relations between the corresponding classical quantities. The simple form of operators corresponding to classical observables is also a great advantage of this representation. These properties of the FW representation ensure its successful use for passing to the semiclassical approximation and finding the classical limit of relativistic quantum mechanics [7, 9]. We note that the Hamiltonian and all other operators are diagonal in two spinors (block-diagonal) in this representation.

When the FW representation is used, the passage to the classical limit is usually accomplished by simply replacing the operators in the expressions for the Hamiltonian and in the operator equations for the dynamics with the corresponding classical quantities. The possibility of such a replacement, explicitly or implicitly used in practically all works devoted to the relativistic FW transformation, was recently rigorously proved in Refs. [10]. This possibility radically simplifies an interpretation of basic quantum-mechanical

equations, especially in the relativistic case.

For a spin-1/2 particle in electromagnetic fields, the initial Hamiltonian can be represented in the general form

$$\mathcal{H}_D = \beta m + \mathcal{E} + \mathcal{O}, \quad \beta \mathcal{E} = \mathcal{E} \beta, \quad \beta \mathcal{O} = -\mathcal{O} \beta, \quad (1)$$

where $\mathcal{E}$ and $\mathcal{O}$ are even and odd operators, respectively, and β is the Dirac matrix. The FW transformation eliminates odd terms in the Hamiltonian.

All old methods of the FW transformation are nonrelativistic. It means that such methods represent the FW Hamiltonian as a series of relativistic corrections in powers of the operators $\mathcal{E}/m$ and $\mathcal{O}/m$. This expansion gives a good solution of the problem for a nonrelativistic particle ($v/c \ll 1$). In particular, it can be used for electrons in atoms (except for heavy atoms) because $|v/c| \sim \sqrt{Z}\alpha \ll 1$. But it does not allow passing to the FW representation for relativistic particles moving in external fields. When $\mathcal{O}^2/m^2 \approx \mathbf{p}^2/m^2 \geq 1$, the series of relativistic corrections does not converge at all. We therefore cannot find compact relativistic expressions for the FW Hamiltonian by the old methods.

The iteration ("step-by-step") method elaborated in Ref. [4] solves this problem. In particular, it allows to obtain relativistic FW Hamiltonians for spin-1/2 particles in electromagnetic fields [4]. However, there is a distinguishing feature of a gravitational interaction of Dirac particles. It consists in a more general form of the initial Hamiltonian. For a Dirac particle in gravitational fields, the mass in Eq. (1) should be replaced with an even operator. Thus, the initial Hamiltonian has the form

$$\mathcal{H}_D = \beta \mathcal{M} + \mathcal{E} + \mathcal{O}, \quad (2)$$

where $\beta \mathcal{M} = \mathcal{M} \beta$.

The general method of the FW transformation of such Hamiltonians has been found in Ref. [5]. It is applicable in the general case of nonstationary external fields and consists in two steps of transformation. The first step is taken with the transformation operator

$$U = \frac{\beta \epsilon + \beta \mathcal{M} - \mathcal{O}}{\sqrt{(\beta \epsilon + \beta \mathcal{M} - \mathcal{O})^2}} \beta, \quad U^{-1} = \beta \frac{\beta \epsilon + \beta \mathcal{M} - \mathcal{O}}{\sqrt{(\beta \epsilon + \beta \mathcal{M} - \mathcal{O})^2}}, \quad (3)$$

where $U^{-1} = U^\dagger$ and $\epsilon = \sqrt{\mathcal{M}^2 + \mathcal{O}^2}$. The transformed Hamiltonian is given by

$$\mathcal{H}' = U \left(\mathcal{H} - i \frac{\partial}{\partial t} \right) U^{-1} + i \frac{\partial}{\partial t}.$$

The exact formula for this Hamiltonian has the form

$$\begin{aligned} \mathcal{H}' = & \beta\epsilon + \mathcal{E} + \frac{1}{2T} \left([T, [T, (\beta\epsilon + \mathcal{E})]] + \beta [\mathcal{O}, [\mathcal{O}, \mathcal{M}]] \right. \\ & - [\mathcal{O}, [\mathcal{O}, \mathcal{E}]] - [(\epsilon + \mathcal{M}), [(\epsilon + \mathcal{M}), \mathcal{E}]] - [(\epsilon + \mathcal{M}), [\mathcal{M}, \mathcal{O}]] \\ & \left. - \beta \{ \mathcal{O}, [(\epsilon + \mathcal{M}), \mathcal{E}] \} + \beta \{ (\epsilon + \mathcal{M}), [\mathcal{O}, \mathcal{E}] \} \right) \frac{1}{T}, \end{aligned} \quad (4)$$

where $T = \sqrt{(\beta\epsilon + \beta\mathcal{M} - \mathcal{O})^2}$.

Hamiltonian (4) still contains odd terms proportional to the first and higher powers of the Planck constant. This Hamiltonian can be presented in the form

$$\mathcal{H}' = \beta\epsilon + \mathcal{E}' + \mathcal{O}', \quad \beta\mathcal{E}' = \mathcal{E}'\beta, \quad \beta\mathcal{O}' = -\mathcal{O}'\beta. \quad (5)$$

The even and odd parts of Hamiltonian (5) are defined by the well-known relations:

$$\mathcal{E}' = \frac{1}{2} (\mathcal{H}' + \beta\mathcal{H}'\beta) - \beta\epsilon, \quad \mathcal{O}' = \frac{1}{2} (\mathcal{H}' - \beta\mathcal{H}'\beta).$$

Additional transformations performed according to Ref. [4] bring $\mathcal{H}'$ to the block-diagonal form. The approximate formula for the final FW Hamiltonian is [5]

$$\mathcal{H}_{FW} = \beta\epsilon + \mathcal{E}' + \frac{1}{4}\beta \left\{ \mathcal{O}'^2, \frac{1}{\epsilon} \right\}. \quad (6)$$

This formula is similar to the corresponding one obtained in Ref. [4].

Equations (4)–(6) solve the problem of the FW transformation for relativistic particles of arbitrary spin in strong external fields.

Any step-by-step methods are approximate [11, 12]. The method developed in Ref. [5] gives exact expressions for terms of the zero and first orders

in the Planck constant $\hbar$ as well as for terms of higher orders in $\hbar$ obtained as a result of the *first* transformation (see Ref. [12]). In practice, exact expressions can also be obtained for leading terms of order of $\hbar^2$ defining the contact (Darwin) gravitational interaction.

Equation (4) can be significantly simplified in some special cases. When $[\mathcal{M}, \mathcal{O}] = 0$, it is reduced to

$$\mathcal{H}' = \beta\epsilon + \mathcal{E} + \frac{1}{2T} \left([T, [T, \mathcal{E}]] - [\mathcal{O}, [\mathcal{O}, \mathcal{E}]] - [(\epsilon + \mathcal{M}), [(\epsilon + \mathcal{M}), \mathcal{E}]] - \beta \{ \mathcal{O}, [(\epsilon + \mathcal{M}), \mathcal{E}] \} + \beta \{ (\epsilon + \mathcal{M}), [\mathcal{O}, \mathcal{E}] \} \right) \frac{1}{T}. \quad (7)$$

In this case, $[\epsilon, \mathcal{M}] = [\epsilon, \mathcal{O}] = 0$ and the operator T is even. It takes the form

$$T = \sqrt{2\epsilon(\epsilon + \mathcal{M})}.$$

3 Spinning particle in curved spacetime

3.1 General spacetime metric

A classical spinning particle is characterized by its position in spacetime, $x^i(\tau)$ which is a function of the proper time τ , and by the 4-vector of spin S^α . The 4-velocity of a particle $U^\alpha = e_i^\alpha dx^i/d\tau$ is normalized by the condition $g_{\alpha\beta} U^\alpha U^\beta = c^2$ where $g_{\alpha\beta} = \text{diag}(c^2, -1, -1, -1)$ is the flat Minkowski metric. To describe spinning particles in flat and curved spacetimes (as well as in arbitrary curvilinear coordinates), we use the tetrad e_i^α . When the gravitational field is absent, one can choose the Cartesian coordinates and apply the coincidence of the holonomic orthonormal frame with the natural one ($e_i^\alpha = \delta_i^\alpha$). For an arbitrary spacetime metric, $g_{\alpha\beta} e_i^\alpha e_j^\beta = g_{ij}$. Let t be the time and x^a ($a = 1, 2, 3$) be spatial local coordinates. The general form of the line element of an arbitrary gravitational field can be given by

$$ds^2 = V^2 c^2 dt^2 - \delta_{\hat{a}\hat{b}} W^{\hat{a}}_c W^{\hat{b}}_d (dx^c - K^c c dt) (dx^d - K^d c dt). \quad (8)$$

The functions V and K^a , as well as the components of the 3×3 matrix $W^{\hat{a}}_{\hat{b}}$ may depend arbitrarily on t, x^a . The total number $1 + 3 + 9 = 13$ of these functions is larger than the number of components of the space-time metric. However, it is obvious that the structure of (8) allows for a redefinition $W^{\hat{a}}_{\hat{b}} \rightarrow L^{\hat{a}}_{\hat{c}} W^{\hat{c}}_{\hat{b}}$ with the help of an arbitrary local rotation $L^{\hat{a}}_{\hat{c}}(t, x) \in SO(3)$. Taking this freedom into account, we end up with exactly 10 independent variables that describe the general spacetime metric.

The off-diagonal metric components g_{0a} are related to the effects of rotation. They arise when the functions K^a are nontrivial. For example, the *exact* metric of the flat spacetime seen by an accelerating and rotating observer is obtained for the special case [13]

$$V = 1 + \frac{\mathbf{a} \cdot \mathbf{r}}{c^2}, \quad W^{\hat{a}}_{\hat{b}} = \delta^a_b, \quad K^a = -\frac{1}{c} (\boldsymbol{\omega} \times \mathbf{r})^a, \quad (9)$$

where $\mathbf{a}$ describes acceleration of the observer and $\boldsymbol{\omega}$ is an angular velocity of a noninertial reference system. Both are independent of the spatial coordinates, but may depend arbitrarily on time t . The slowly rotating massive body produces the gravitational field of similar form; far from the source the Lense-Thirring [14] metric is described by

$$V = V(x), \quad W^{\hat{a}}_{\hat{b}} = \delta^a_b W(x), \quad K^a = \frac{1}{c} \epsilon^{abc} \omega_b(x) x_c. \quad (10)$$

The functions $V(x^a), W(x^a), \omega(x^a)$ can be recovered from the Kerr metric in the isotropic coordinates in the limit of $r \rightarrow \infty$ (see Ref. [15]):

$$V = \left(1 - \frac{\mu}{2r}\right) \left(1 + \frac{\mu}{2r}\right)^{-1} - \frac{\mu a^2 - 3\mu(\mathbf{a} \cdot \mathbf{n})^2}{2r^3} + \mathcal{O}(a^2 r^{-4}), \quad (11)$$

$$W = \left(1 + \frac{\mu}{2r}\right)^2 + \frac{\mu a^2 - 3\mu(\mathbf{a} \cdot \mathbf{n})^2}{2r^3} + \mathcal{O}(a^2 r^{-4}), \quad (12)$$

and K^a is given by the same equation (10) with

$$\boldsymbol{\omega} = \frac{2\mu c}{r^3} \mathbf{a} \left(1 - \frac{3\mu}{r} + \frac{21\mu^2}{4r^2}\right) + \mathcal{O}(a^3 r^{-5}). \quad (13)$$

Here $r := \sqrt{\mathbf{x} \cdot \mathbf{x}}$ and $\mathbf{n} = \mathbf{r}/r$. The constant vector $\mathbf{a} = (0, 0, a)$ is the rotation parameter of the Kerr solution and $\mathbf{a} \cdot \mathbf{n} = az/r$. Also, $\mu = GM/c^2$;

the total mass M and the total angular momentum $\mathbf{J} = M\mathbf{ca}$ define the Kerr black hole uniquely. These equations are obtained from the Arnowitt-Deser-Misner form [16] of the Kerr solution performed earlier by Hergt and Schäfer [17] after dropping the terms violating the isotropy.

In the weak field approximation, the dynamics of quantum and classical spin have been previously studied in Ref. [19] for the cases (9) and (10).

3.2 Dirac fermions

In order to discuss the Dirac spinors, we need the orthonormal frames. The preferable choice [18, 19] is the Schwinger gauge:

$$e_i^{\hat{0}} = V \delta_i^0, \quad e_i^{\hat{a}} = W^{\hat{a}}_b (\delta_i^b - cK^b \delta_i^0), \quad a = 1, 2, 3. \quad (14)$$

Tetrad (14) is characterized by the condition $e_a^{\hat{0}} = 0$, $a = 1, 2, 3$. The same is automatically true for the inverse tetrad:

$$e_{\hat{0}}^i = \frac{1}{V} (\delta_0^i + \delta_a^i cK^a), \quad e_{\hat{a}}^i = \delta_b^i W^b_{\hat{a}}, \quad a = 1, 2, 3, \quad (15)$$

where the inverse 3×3 matrix, $W^a_{\hat{c}} W^{\hat{c}}_b = \delta_b^a$, is introduced.

The covariant Dirac equation for spin-1/2 particles in gravitational and electromagnetic fields has the form

$$(i\hbar\gamma^\alpha D_\alpha - mc)\Psi = 0, \quad \alpha = 0, 1, 2, 3. \quad (16)$$

The Dirac matrices γ^α are defined in local Lorentz (tetrad) frames. The spinor covariant derivatives are given by

$$D_\alpha = e_\alpha^i D_i, \quad D_i = \partial_i + \frac{iq}{\hbar c} A_i + \frac{i}{4} \sigma^{\alpha\beta} \Gamma_{i\alpha\beta}, \quad \sigma^{\alpha\beta} = \frac{i}{2} (\gamma^\alpha \gamma^\beta - \gamma^\beta \gamma^\alpha). \quad (17)$$

Here $\Gamma_i^{\alpha\beta} = -\Gamma_i^{\beta\alpha}$ are the Lorentz connection coefficients and A_i is the 4-potential of the electromagnetic field. The Dirac particle is characterised by the electric charge q .

Eqs. (16),(17) show that the gravitational and inertial effects are encoded in coframe and connection (see the relevant discussion in Refs. [20, 21])

and references therein). For the general metric (8) with the tetrad (14) we find explicitly [15]

$$\Gamma_{i\hat{a}\hat{0}} = \frac{c^2}{V} W^b_{\hat{a}} \partial_b V e_i^{\hat{0}} - \frac{c}{V} Q_{(\hat{a}\hat{b})} e_i^{\hat{b}}, \quad (18)$$

$$\Gamma_{i\hat{a}\hat{b}} = \frac{c}{V} Q_{[\hat{a}\hat{b}]} e_i^{\hat{0}} + (C_{\hat{a}\hat{b}\hat{c}} + C_{\hat{a}\hat{c}\hat{b}} + C_{\hat{c}\hat{b}\hat{a}}) e_i^{\hat{c}}. \quad (19)$$

Here

$$Q_{\hat{a}\hat{b}} = g_{\hat{a}\hat{c}} W^d_{\hat{b}} \left(\frac{1}{c} \dot{W}^{\hat{c}}_d + K^e \partial_e W^{\hat{c}}_d + W^{\hat{c}}_e \partial_d K^e \right), \quad (20)$$

$$C_{\hat{a}\hat{b}}^{\hat{c}} = W^d_{\hat{a}} W^e_{\hat{b}} \partial_{[d} W^{\hat{c}}_{e]} = -C_{\hat{b}\hat{a}}^{\hat{c}}, \quad C_{\hat{a}\hat{b}\hat{c}} = g_{\hat{c}\hat{d}} C_{\hat{a}\hat{b}}^{\hat{d}}. \quad (21)$$

The dot $\dot{}$ denotes the derivative with respect to the time t . As one notices, $C_{\hat{a}\hat{b}}^{\hat{c}}$ is nothing but the anholonomy object for the spatial triad $W^{\hat{a}}_{\hat{b}}$. The indices (that all run from 1 to 3) are raised and lowered with the help of the spatial part of the flat Minkowski metric, $g_{\hat{a}\hat{b}} = -\delta_{ab} = \text{diag}(-1, -1, -1)$.

The Schrödinger-like equation derived from the Dirac one has a non-Hermitian Hamiltonian. This problem disappears if we define a new wave function by [15]

$$\psi = (\sqrt{-g}e_0^0)^{\frac{1}{2}} \Psi. \quad (22)$$

Such a non-unitary transformation may be also interpreted in the framework of the pseudo-Hermitian quantum mechanics [22, 23] (cf. also [24]).

Hereinafter, the denotations $V = e_0^{\hat{0}}$, $\mathcal{F}^b_a = \sqrt{-g}e_a^b = VW^b_{\hat{a}}$, and

$$\Upsilon = -V\epsilon^{\hat{a}\hat{b}\hat{c}}\Gamma_{\hat{a}\hat{b}\hat{c}} = -V\epsilon^{\hat{a}\hat{b}\hat{c}}C_{\hat{a}\hat{b}\hat{c}}, \quad \Xi_{\hat{a}} = \frac{V}{c}\epsilon_{\hat{a}\hat{b}\hat{c}}\Gamma_{\hat{0}}^{\hat{b}\hat{c}} = \epsilon_{\hat{a}\hat{b}\hat{c}}Q^{\hat{b}\hat{c}} \quad (23)$$

will be used. For the static and stationary rotating configurations, the pseudoscalar invariant vanishes ($\epsilon^{\hat{a}\hat{b}\hat{c}}C_{\hat{a}\hat{b}\hat{c}} = 0$), and thus the corresponding term was absent in the special cases considered in Ref. [19]. But in general this term contributes to the Dirac Hamiltonian.

The *Hermitian* Hamiltonian in the Schrödinger-like equation $i\hbar\frac{\partial\psi}{\partial t} = \mathcal{H}\psi$ reads [15]

$$\begin{aligned} \mathcal{H} = & \beta mc^2 V + q\Phi + \frac{c}{2} (\pi_b \mathcal{F}^b_a \alpha^a + \alpha^a \mathcal{F}^b_a \pi_b) \\ & + \frac{c}{2} (\mathbf{K} \cdot \boldsymbol{\pi} + \boldsymbol{\pi} \cdot \mathbf{K}) + \frac{\hbar c}{4} (\Upsilon \gamma_5 + \boldsymbol{\Xi} \cdot \boldsymbol{\Sigma}). \end{aligned} \quad (24)$$

The kinetic momentum operator $\pi_i = i\hbar\partial_i - \frac{q}{c}A_i = p_i - \frac{q}{c}A_i$ accounts of the interaction with the electromagnetic field $A_i = (\Phi, A_a)$. Hamiltonian (24) covers the general case of a spin-1/2 particle in an arbitrary curved spacetime.

Since the metric tensor is symmetric, it can be brought to the form diagonal *in spatial coordinates*. For example, the Kerr metric in the both spherical and Boyer-Lindquist coordinates belongs to this form.

A spatially diagonal metric tensor can often be reduced to an isotropic form by an appropriate transformation of *spatial coordinates*. In spatially isotropic coordinates, $W^{\hat{a}}_{\hat{b}} = W\delta^{\hat{a}}_{\hat{b}}$ and the final form of the line element is defined by

$$ds^2 = V^2 c^2 dt^2 - W^2 \delta_{ab} (dx^a - K^a c dt) (dx^b - K^b c dt). \quad (25)$$

Evidently, none of the two transformations changes the temporal coordinate. Since only the spatial coordinates are transformed, the transformations do not change the physical frame (particle rest frame) used for a definition of the three-component physical spin.

For isotropic metric (25) with the Schwinger gauge, the exact Hermitian Dirac Hamiltonian is given by [19]

$$\begin{aligned} \mathcal{H} = & \beta mc^2 V + \frac{c}{2} [(\boldsymbol{\alpha} \cdot \mathbf{p}) \mathcal{F} + \mathcal{F}(\boldsymbol{\alpha} \cdot \mathbf{p})] \\ & + \frac{c}{2} (\mathbf{K} \cdot \mathbf{p} + \mathbf{p} \cdot \mathbf{K}) + \frac{\hbar c}{4} (\boldsymbol{\nabla} \times \mathbf{K}) \cdot \boldsymbol{\Sigma}, \end{aligned} \quad (26)$$

where

$$\mathcal{F} = V/W. \quad (27)$$

It covers the general case of a spin-1/2 particle in an isotropic stationary metric.

4 The Foldy-Wouthuysen representation

Let us now derive the FW Hamiltonian for stationary spatially isotropic metric (25). We perform the FW transformation of the exact Dirac Hamiltonian (26) with the help of the method developed in Ref. [5]. Since the

gravitational field is supposed to be strong, we do not make any approximations for the functions V, W, K^a , and the only small parameter is the Planck constant $\hbar$. In the FW Hamiltonian, we retain all terms of the zero and first orders in $\hbar$ and leading terms of order of $\hbar^2$ nonvanishing in the both nonrelativistic and weak field approximations. These terms describe the gravitational contact (Darwin) interaction [25]. The computations are straightforward, and the final FW Hamiltonian is given by [15]

$$\mathcal{H}_{FW} = \mathcal{H}_{FW}^{(1)} + \mathcal{H}_{FW}^{(2)}, \quad (28)$$

where

$$\begin{aligned} \mathcal{H}_{FW}^{(1)} = & \beta\epsilon' - \beta \frac{\hbar m c^4}{4} \left\{ \frac{1}{2\epsilon'^2 + mc^2\{\epsilon', V\}}, \left[\Sigma \cdot (\Phi \times p) - \Sigma \cdot (p \times \Phi) \right. \right. \\ & \left. \left. + \hbar \nabla \cdot \Phi \right] \right\} + \beta \frac{\hbar c^2}{16} \left\{ \frac{1}{\epsilon'}, \left[\Sigma \cdot (\mathcal{G} \times p) - \Sigma \cdot (p \times \mathcal{G}) + \hbar \nabla \cdot \mathcal{G} \right] \right\}, \end{aligned} \quad (29)$$

$$\begin{aligned} \mathcal{H}_{FW}^{(2)} = & \frac{c}{2} (K \cdot p + p \cdot K) + \frac{\hbar c}{4} \Sigma \cdot (\nabla \times K) \\ & - \frac{\hbar c^3}{16} \left\{ \frac{1}{2\epsilon'^2 + mc^2\{\epsilon', V\}}, \{\mathcal{F}^2, \Sigma \cdot Q\} \right\}, \end{aligned} \quad (30)$$

$$\epsilon' = \sqrt{m^2 c^4 V^2 + \frac{1}{2} c^2 \{\mathcal{F}^2, p^2\}}, \quad \mathcal{G} = \nabla(\mathcal{F}^2), \quad \Phi = \mathcal{F}^2 \nabla V, \quad (31)$$

$$\begin{aligned} Q = & p \times \nabla (K \cdot p + p \cdot K) - \nabla (K \cdot p + p \cdot K) \times p \\ & - p \times (p \times (\nabla \times K)) - ((\nabla \times K) \times p) \times p. \end{aligned} \quad (32)$$

The equivalent forms of the latter quantity

$$\begin{aligned} Q = & p \times (p \times (\nabla \times K)) + ((\nabla \times K) \times p) \times p \\ & + 2p \times (p \cdot \nabla) K - 2\delta^{ab} (\nabla_a K) p_b \times p, \\ Q = & p \times \nabla (p \cdot K) - \nabla (K \cdot p) \times p + p \times (p \cdot \nabla) K - \delta^{ab} (\nabla_a K) p_b \times p \end{aligned} \quad (33)$$

may be also useful.

The dynamical equation for the spin obtained from the commutator of the FW Hamiltonian with the polarization operator $\mathbf{\Pi} = \beta \mathbf{\Sigma}$ is given by

$$\frac{d\mathbf{\Pi}}{dt} = \frac{i}{\hbar} [\mathcal{H}_{FW}, \mathbf{\Pi}] = \mathbf{\Omega}^{(1)} \times \mathbf{\Sigma} + \mathbf{\Omega}^{(2)} \times \mathbf{\Pi}, \quad (34)$$

where $\mathbf{\Omega}^{(1)}$ is the operator of angular velocity of rotation of the spin in the static gravitational field,

$$\begin{aligned} \mathbf{\Omega}^{(1)} = & -\frac{mc^4}{2} \left\{ \frac{1}{2\epsilon'^2 + mc^2\{\epsilon', V\}}, (\mathbf{\Phi} \times \mathbf{p} - \mathbf{p} \times \mathbf{\Phi}) \right\} \\ & + \frac{c^2}{8} \left\{ \frac{1}{\epsilon'}, (\mathbf{g} \times \mathbf{p} - \mathbf{p} \times \mathbf{g}) \right\}, \end{aligned} \quad (35)$$

and the contribution from the nondiagonal part of the metric is equal to

$$\mathbf{\Omega}^{(2)} = \frac{c}{2} \nabla \times \mathbf{K} - \frac{c^3}{8} \left\{ \frac{1}{2\epsilon'^2 + mc^2\{\epsilon', V\}}, \{\mathcal{F}^2, \mathbf{Q}\} \right\}. \quad (36)$$

The two different matrices appear on the right-hand side of Eq. (34) due to the fact that $\mathbf{\Omega}^{(1)}$ contains the velocity operator rather than the momentum one. Since the velocity operator is proportional to an additional β factor and is equal to $\mathbf{v} = \beta c \mathbf{p} / \epsilon$ for free particles, the operator $\mathbf{\Omega}^{(1)}$, when expressed in terms of $\mathbf{v}$, also acquires an additional β factor [19].

The corresponding semiclassical formulas describing the motion of the average spin are then explicitly given by

$$\frac{d\mathbf{s}}{dt} = \mathbf{\Omega} \times \mathbf{s} = (\mathbf{\Omega}^{(1)} + \mathbf{\Omega}^{(2)}) \times \mathbf{s}, \quad (37)$$

$$\mathbf{\Omega}^{(1)} = \frac{mc^4}{\epsilon'(\epsilon' + mc^2V)} \mathbf{p} \times \mathbf{\Phi} - \frac{c^2}{2\epsilon'} \mathbf{p} \times \mathbf{g}, \quad (38)$$

$$\mathbf{\Omega}^{(2)} = \frac{c}{2} \nabla \times \mathbf{K} - \frac{c}{4\epsilon'(\epsilon' + mc^2V)} \mathcal{F}^2 \mathbf{Q}, \quad (39)$$

where, in the semiclassical limit,

$$\epsilon' = \sqrt{m^2 c^4 V^2 + c^2 \mathbf{p}^2 \mathcal{F}^2}, \quad \mathbf{Q} = 2\mathbf{p} \times \nabla(\mathbf{p} \cdot \mathbf{K}) + 2\mathbf{p} \times (\mathbf{p} \cdot \nabla) \mathbf{K}. \quad (40)$$

FW Hamiltonian (28) can now be expressed in the simpler form in terms of $\Omega^{(1)}, \Omega^{(2)}$ [15]:

$$\mathcal{H}_{FW} = \beta\epsilon' + \frac{c}{2}(\mathbf{K} \cdot \mathbf{p} + \mathbf{p} \cdot \mathbf{K}) + \frac{\hbar}{2}\mathbf{\Pi} \cdot \mathbf{\Omega}^{(1)} + \frac{\hbar}{2}\mathbf{\Sigma} \cdot \mathbf{\Omega}^{(2)} - \beta \frac{\hbar^2 m c^4}{4} \left\{ \frac{1}{2\epsilon'^2 + m c^2 \{\epsilon', V\}}, \nabla \cdot \Phi \right\} + \beta \frac{\hbar^2 c^2}{16} \left\{ \frac{1}{\epsilon'}, \nabla \cdot \mathcal{G} \right\}. \quad (41)$$

It is instructive to compare the classical and quantum Hamiltonians of a spinning particle. In order to do this, one can start from the classical Hamiltonian of a spinless particle [26]:

$$\mathcal{H}_{class} = \left(\frac{m^2 c^2 - \tilde{g}^{ab} \pi_a \pi_b}{g^{00}} \right)^{1/2} + \frac{g^{0a} \pi_a}{g^{00}} + q A_0, \quad \tilde{g}^{ab} = g^{ab} - \frac{g^{0a} g^{0b}}{g^{00}}. \quad (42)$$

Substituting the components of the general metric (8), we recast this into

$$\mathcal{H}_{class} = \sqrt{m^2 c^4 V^2 + c^2 \delta^{cd} \mathcal{F}_c^a \mathcal{F}_d^b \pi_a \pi_b} + c \mathbf{K} \cdot \boldsymbol{\pi} + q \Phi. \quad (43)$$

In order to take into account the spin correctly, this Hamiltonian should be completed by the interaction term $\mathbf{s} \cdot \boldsymbol{\Omega}$, along the lines of the general discussion of the Ref. [27]:

$$\mathcal{H}_{class} = \sqrt{m^2 c^4 V^2 + c^2 \delta^{cd} \mathcal{F}_c^a \mathcal{F}_d^b \pi_a \pi_b} + c \mathbf{K} \cdot \boldsymbol{\pi} + q \Phi + \mathbf{s} \cdot \boldsymbol{\Omega}. \quad (44)$$

In the general case, $\boldsymbol{\Omega}$ includes the both electromagnetic and gravitational contributions.

For the case of the stationary spatially isotropic metric (25), we obtain the resulting classical Hamiltonian (equal just to $c p_0$)

$$\mathcal{H}_{class} = \sqrt{m^2 c^4 V^2 + c^2 \mathcal{F}^2 \mathbf{p}^2} + c \mathbf{K} \cdot \mathbf{p} + \mathbf{s} \cdot \boldsymbol{\Omega}, \quad (45)$$

where terms dependent on the electromagnetic fields are omitted. It is satisfactory to notice that the quantum Hamiltonians (28) and (41) completely agree with the classical Hamiltonian (45).

The similarity of the quantum and the classical Hamiltonians naturally leads to the similarity of the quantum and classical equations of motion discussed in the next section.

5 Classical dynamics of spinning particles

The dynamics of spinning particles in strong gravitational fields is described by the generally covariant Mathisson-Papapetrou [28, 29] theory. A different approach was developed by Khriplovich and Pomeransky [27] for the noncovariant 3-dimensional spin defined in the particle rest frame. In general, the analysis of the Mathisson-Papapetrou equations is a difficult task and various approximation schemes were developed for their solution. By neglecting the second order spin effects, the Mathisson-Papapetrou system is reduced to [30]

$$\frac{DU^\alpha}{d\tau} = f_m^\alpha = -\frac{1}{2m} S^{\mu\nu} U^\beta R_{\mu\nu\beta}{}^\alpha, \quad (46)$$

$$\frac{DS^\alpha}{d\tau} = 0. \quad (47)$$

The Mathisson force f_m^α in the right-hand side of (46) depends on the curvature $R_{\mu\nu\beta}{}^\alpha$ of spacetime. The physical spin $\mathbf{s}$ is defined in the rest frame of a particle. Taking into account that the 4-velocity is

$$U^i = \gamma(e_0^i + v^a e_a^i), \quad (48)$$

we can recast the Mathisson-Papapetrou system into the 3-vector form

$$\frac{d\mathbf{v}}{d\tau} = \boldsymbol{\mathcal{E}} - \frac{\mathbf{v}(\mathbf{v} \cdot \boldsymbol{\mathcal{E}})}{c^2} + \mathbf{v} \times \boldsymbol{\mathcal{B}} + \frac{1}{\gamma} \mathbf{f}_m, \quad (49)$$

$$\frac{d\mathbf{s}}{d\tau} = \boldsymbol{\Omega} \times \mathbf{s}, \quad \boldsymbol{\Omega} = -\boldsymbol{\mathcal{B}} + \frac{\gamma}{\gamma+1} \frac{\mathbf{v} \times \boldsymbol{\mathcal{E}}}{c^2}, \quad (50)$$

by introducing the fields $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{B}}$ [27],

$$-U^i \Gamma_{i\hat{0}}{}^{\hat{a}} = \mathcal{E}^a, \quad -U^i \Gamma_{i\hat{b}}{}^{\hat{a}} = \epsilon^a{}_{bc} \mathcal{B}^c. \quad (51)$$

This represents a clear analogy between the gravitational and electromagnetic fields that makes it possible to speak of the gravitoelectromagnetic type effects. One should not confuse the objects (51) with the usual gravitoelectromagnetic fields [31, 32] that are defined in the weak-field approximation and that satisfy the Maxwell-like dynamical equations which are

derived from Einstein's gravitational equation. In contrast, the fields (51) arise in a purely kinematic context and are defined for arbitrarily strong field configurations. Note also that they depend on the velocity of the particle, unlike the usual gravitoelectromagnetic fields [31, 32]. Nevertheless, their application is quite helpful because the equations of motion of particles and their spins, written in terms of $\mathcal{E}$ and $\mathcal{B}$, look very similar to the corresponding equations of motion of charged spinning particles in the electrodynamics.

Let us calculate these fields explicitly. We start from Eqs. (18) and (19) and use the notation from Ref. [19]. Multiplying the connection coefficients (18)-(19) by the 4-velocity (48), we find:

$$U^i \Gamma_{i\hat{a}\hat{b}} = \frac{c\gamma}{V} \mathcal{Q}_{[\hat{a}\hat{b}]} + \gamma (C_{\hat{a}\hat{b}\hat{c}} + C_{\hat{a}\hat{c}\hat{b}} + C_{\hat{c}\hat{b}\hat{a}}) v^{\hat{c}}, \quad (52)$$

$$U^i \Gamma_{i\hat{0}\hat{b}} = \frac{c\gamma}{V} \mathcal{Q}_{(\hat{b}\hat{c})} v^{\hat{c}} - \frac{c^2\gamma}{V} W^c_{\hat{b}} \partial_c V. \quad (53)$$

As a result, for the general metric (8), the gravitoelectric and gravitomagnetic fields (51) read [15]

$$\mathcal{E}_a = \frac{\gamma}{V} \left(c \mathcal{Q}_{(\hat{a}\hat{b})} v^b - c^2 W^b_{\hat{a}} \partial_b V \right), \quad (54)$$

$$\mathcal{B}^a = \frac{\gamma}{V} \left(-\frac{c}{2} \Xi^a - \frac{1}{2} \Upsilon v^a + \epsilon^{abc} V C_{bc}{}^d v_d \right). \quad (55)$$

The physical spin precesses (50) with the angular velocity Ω that can also be written explicitly in terms of the connection [19, 27]

$$\Omega_{\hat{a}} = \epsilon_{abc} U^i \left(\frac{1}{2} \Gamma_i{}^{\hat{c}\hat{b}} + \frac{\gamma}{\gamma+1} \Gamma_{i\hat{0}}{}^{\hat{b}} v^{\hat{c}} / c^2 \right). \quad (56)$$

As compared with the quantities $\Omega^{(1)}$ and $\Omega^{(2)}$ describing the precession of the quantum spin using the coordinate time, $\Omega_{\hat{a}}$ contains an extra factor $dt/d\tau = U^0 = \gamma/V$ in view of a different parameterization using the proper time.

Let us calculate the classical precession angular velocity in the Schwinger gauge explicitly. Substituting (52) and (53) into (56), we obtain the *exact*

classical formula for the angular velocity of the spin precession in an arbitrary gravitational field [15]:

$$\begin{aligned}\Omega^{\hat{a}} = & \frac{\gamma}{V} \left(\frac{1}{2} \Upsilon v^{\hat{a}} - \epsilon^{abc} V \mathcal{C}_{\hat{b}\hat{c}}{}^d v_{\hat{d}} + \frac{\gamma}{\gamma+1} \epsilon^{abc} W^d_{\hat{b}} \partial_d V v_{\hat{c}} \right. \\ & \left. + \frac{c}{2} \Xi^{\hat{a}} - \frac{\gamma}{\gamma+1} \epsilon^{abc} \mathcal{Q}_{(\hat{b}\hat{d})} \frac{v^{\hat{d}} v_{\hat{c}}}{c} \right). \quad (57)\end{aligned}$$

The terms in the first line are linear in the 4-velocity of the particle, whereas the terms in the second line contain the even number of the velocity factors. For the special class of stationary geometries (25), this general formula reduces to the simpler expression [15]:

$$\begin{aligned}\Omega = & \frac{\gamma}{V} \left[-V \mathbf{v} \times \nabla \left(\frac{1}{W} \right) - \frac{\gamma}{W(\gamma+1)} \mathbf{v} \times \nabla V \right. \\ & \left. + \frac{c}{2} \nabla \times \mathbf{K} - \frac{\gamma}{2c(\gamma+1)} (\mathbf{v} \times \nabla (\mathbf{v} \cdot \mathbf{K}) + \mathbf{v} \times (\mathbf{v} \cdot \nabla) \mathbf{K}) \right]. \quad (58)\end{aligned}$$

Equivalently,

$$\begin{aligned}\Omega = & \frac{\gamma}{V} \left[\frac{1}{\mathcal{F}V(\gamma+1)} \mathbf{v} \times \Phi - \frac{1}{2\mathcal{F}} \mathbf{v} \times \mathcal{G} \right. \\ & \left. + \frac{c}{2} \nabla \times \mathbf{K} - \frac{\gamma}{2c(\gamma+1)} (\mathbf{v} \times \nabla (\mathbf{v} \cdot \mathbf{K}) + \mathbf{v} \times (\mathbf{v} \cdot \nabla) \mathbf{K}) \right]. \quad (59)\end{aligned}$$

Since

$$\epsilon' = mc^2 V \sqrt{1 + \frac{\mathbf{p}^2}{W^2 m^2 c^2}} = mc^2 V \gamma, \quad \mathbf{p} = mW \gamma \mathbf{v}, \quad (60)$$

we can conclude that the classical equation of the spin motion (50) agrees with the quantum equation (34) and with the semiclassical one (37). Thus, the classical and the quantum theories of the spin motion in gravity are in complete agreement. This is now verified for the arbitrary strong field configurations.

6 Evolution of helicity in an isotropic metric

Let us analyse the semiclassical evolution of the helicity of a particle propagating in a strong gravitation field. The helicity describes the spin orientation with respect to the direction of particle's motion. As usual, one means the orientation of the 3-component spin defined in the particle rest frame (physical spin). The particle motion is defined by the trajectory that shows how the contravariant spatial world coordinates of the particle change in time. Evidently, the particle motion can be correctly characterized by the evolution of the *contravariant* world velocity or the unit vector in its direction. As a result, one can unambiguously define the helicity as a projection of the 3-component spin (pseudo)vector *in the particle rest frame* onto the direction of the unit vector along the contravariant velocity *in the world frame*. Thus, the helicity should be defined as

$$\zeta = (\mathbf{s}/s) \cdot (\mathbf{U}/U) = (\mathbf{s}/s) \cdot \mathbf{V}/V,$$

where $\mathbf{U} = \{U^1, U^2, U^3\}$ and $\mathbf{V} = \mathbf{U}/U^0$. The investigation of the helicity is simplified when particle's trajectory is infinite. In this case, one can apply the fact that the vector $\mathbf{N} = \mathbf{U}/U$ coincides with the vectors $\mathbf{v}/v$ and $\mathbf{p}/p$ at the initial and final parts of such a trajectory of the particle because of the very large distance to the field source. Here $\mathbf{v} = \{v^{\hat{1}}, v^{\hat{2}}, v^{\hat{3}}\}$ is the velocity in the anholonomic *coframe* (48) and $\mathbf{p} = \{-p_1, -p_2, -p_3\}$ is the *covariant* momentum entering the classical and quantum Hamiltonians. For the isotropic metric under consideration, $p_a = e_a^{\hat{0}} p_{\hat{0}} + e_a^{\hat{b}} p_{\hat{b}} = W p_{\hat{a}}$ and $p_a/p = W v_{\hat{a}}/(|W|v) = v_{\hat{a}}/v$. As a result, the change of the helicity on the whole trajectory can be given by $\Delta\zeta' = \Delta\zeta$, where $\zeta' = (\mathbf{s}/s) \cdot \mathbf{n}$, $\mathbf{n} = \mathbf{v}/v = \mathbf{p}/p$. Evidently, $\zeta' = \cos \chi$, where χ is an angle between the $\mathbf{s}$ and $\mathbf{n}$ vectors.

In the general case, the evolution of the covariant momentum operator is defined by

$$\frac{d\mathbf{p}}{dt} = \frac{i}{\hbar} [\mathcal{H}_{FW}, \mathbf{p}]. \quad (61)$$

In the discussion of the semiclassical dynamics of the particle, we can neglect the small spin-dependent force which enters into the equation of motion with

an additional factor $\hbar$. Thus, in the semiclassical approximation

$$\frac{dp^{-1}}{dt} = -\frac{1}{p^3} \mathbf{p} \cdot \frac{d\mathbf{p}}{dt},$$

and the dynamics of the unit vector $\mathbf{n}$ is given by

$$\frac{d\mathbf{n}}{dt} = -\frac{1}{p} \mathbf{n} \times \left(\mathbf{n} \times \frac{d\mathbf{p}}{dt} \right). \quad (62)$$

Therefore, the vector $\mathbf{n}$ rotates with the angular velocity

$$\tilde{\omega} = \frac{1}{p} \mathbf{n} \times \frac{d\mathbf{p}}{dt}, \quad (63)$$

and one can add a quantity $\kappa \mathbf{n}$ with an arbitrary factor κ . Using the Eqs. (31), (41), (60), (61), (63), we derive

$$\tilde{\omega} = -\frac{c^2}{\mathcal{F}V\gamma^2v} \mathbf{n} \times \Phi - \frac{v}{2\mathcal{F}} \mathbf{n} \times \mathcal{G} - c\mathbf{n} \times \nabla(\mathbf{n} \cdot \mathbf{K}). \quad (64)$$

The nabla operator does not act on $\mathbf{n}$.

We can add the quantity $\frac{c}{2} \mathbf{n}(\mathbf{n} \cdot (\mathbf{n} \times \mathbf{K}))$ to $\tilde{\omega}$ and use the identity

$$2\mathbf{n} \times \nabla(\mathbf{n} \cdot \mathbf{K}) = \mathbf{n}(\mathbf{n} \cdot (\mathbf{n} \times \mathbf{K})) - \nabla \times \mathbf{K} + \mathbf{n} \times \nabla(\mathbf{n} \cdot \mathbf{K}) + \mathbf{n} \times (\mathbf{n} \cdot \nabla) \mathbf{K}.$$

As a result, Eq. (64) is recast into [15]

$$\begin{aligned} \tilde{\omega} = & -\frac{c^2}{\mathcal{F}V\gamma^2v} \mathbf{n} \times \Phi - \frac{v}{2\mathcal{F}} \mathbf{n} \times \mathcal{G} + \frac{c}{2} \nabla \times \mathbf{K} \\ & - \frac{c}{2} [\mathbf{n} \times \nabla(\mathbf{n} \cdot \mathbf{K}) + \mathbf{n} \times (\mathbf{n} \cdot \nabla) \mathbf{K}]. \end{aligned} \quad (65)$$

The angular velocity of the spin rotation defined by Eq. (39) can be expressed in the form similar to Eq. (59):

$$\begin{aligned} \Omega = & \frac{v}{\mathcal{F}V(\gamma+1)} \mathbf{n} \times \Phi - \frac{v}{2\mathcal{F}} \mathbf{n} \times \mathcal{G} \\ & + \frac{c}{2} \nabla \times \mathbf{K} - \frac{c(\gamma-1)}{2\gamma} [\mathbf{n} \times \nabla(\mathbf{n} \cdot \mathbf{K}) + \mathbf{n} \times (\mathbf{n} \cdot \nabla) \mathbf{K}]. \end{aligned} \quad (66)$$

Therefore, the vector of spin rotates with respect to the momentum direction, and the angular velocity of such rotation is [15]

$$\boldsymbol{o} = \boldsymbol{\Omega} - \tilde{\boldsymbol{\omega}} = \frac{c^2}{\mathcal{F}V\gamma v} \boldsymbol{n} \times \boldsymbol{\Phi} + \frac{c}{2\gamma} [\boldsymbol{n} \times \nabla(\boldsymbol{n} \cdot \boldsymbol{K}) + \boldsymbol{n} \times (\boldsymbol{n} \cdot \nabla)\boldsymbol{K}]. \quad (67)$$

The same result can be obtained for an arbitrary metric by the purely classical method considered in the previous section. Indeed, let us use classical equations (49),(50) and neglect the relatively small Mathisson force $\boldsymbol{f}_m$. Since

$$\frac{d\boldsymbol{n}}{d\tau} = -\frac{1}{v} \boldsymbol{n} \times \left(\boldsymbol{n} \times \frac{d\boldsymbol{v}}{d\tau} \right), \quad (68)$$

Eq. (49) leads to the following formula:

$$\frac{d\boldsymbol{n}}{d\tau} = \boldsymbol{\omega}' \times \boldsymbol{n}, \quad (69)$$

where

$$\boldsymbol{\omega}' = -\boldsymbol{\mathcal{B}} + \frac{1}{v^2} \boldsymbol{v} \times \boldsymbol{\mathcal{E}}. \quad (70)$$

Consequently, we find that the spin vector rotates with respect to the momentum direction and the angular velocity of this rotation is

$$\boldsymbol{o} = \frac{V}{\gamma} (\boldsymbol{\Omega} - \boldsymbol{\omega}') = -V \frac{\boldsymbol{v} \times \boldsymbol{\mathcal{E}}}{\gamma^2 v^2}. \quad (71)$$

The gravitomagnetic field $\boldsymbol{\mathcal{B}}$ does not influence particle's helicity [33]. This conclusion is valid for an arbitrary gravitational field when the effect of the Mathisson force on the particle motion is neglected. $\boldsymbol{\mathcal{E}}$ is proportional to γ/V . The use of the world time shows that any gravitational field does not affect the helicity of a particle with a negligible mass.

The covariant Dirac equation can be used and the FW transformation can be performed for a massless particle. However, the interpretation of the results is a serious problem. The 3-component spin is defined in the particle rest frame which cannot be introduced for such a particle.

For stationary isotropic metric (25), the gravitoelectric and the gravitomagnetic fields (54) and (55) reduce to

$$\mathcal{E}_a = -\frac{\gamma c^2}{V} \left(\frac{1}{W} \partial_a V + \frac{K^b \partial_b W}{W} \frac{v_a}{c} + \partial_{(a} K_{b)} \frac{v^b}{c} \right), \quad (72)$$

$$\mathcal{B}^a = \frac{\gamma}{V} \left(-\frac{c}{2} \epsilon^{abc} \partial_b K_c + \epsilon^{abc} V \frac{\partial_b W}{W^2} v_c \right). \quad (73)$$

This demonstrates the perfect agreement between the quantum and classical analyses, cf. Eqs. (67) and (71).

As a final remark, we should mention that the results presented in this section cannot, generally speaking, be applied for the particle moving on a finite trajectory because the replacement of ζ by ζ' is not always possible in this case [15].

6.1 General noninertial frame

A general noninertial frame is an important application of the results above. This frame is characterized by the acceleration $\mathbf{a}$ and the rotation $\boldsymbol{\omega}$ of an observer. The *exact* metric of the flat spacetime seen by the accelerated and rotating observer has the form (25) and is given by Eq. (9).

We can apply our general results (34)–(36), (41) for the metric (9) of the flat spacetime. For the FW Hamiltonian we then find

$$\begin{aligned} \mathcal{H}_{FW} &= \mathcal{H}_0 + \frac{\hbar}{2} \boldsymbol{\Pi} \cdot \boldsymbol{\Omega}^{(1)} + \frac{\hbar}{2} \boldsymbol{\Sigma} \cdot \boldsymbol{\Omega}^{(2)}, \\ \mathcal{H}_0 &= \frac{\beta}{2} \left\{ \left(1 + \frac{\mathbf{a} \cdot \mathbf{r}}{c^2} \right), \sqrt{m^2 c^4 + c^2 \mathbf{p}^2} \right\} - \boldsymbol{\omega} \cdot \mathbf{l}, \\ \boldsymbol{\Omega}^{(1)} &= \frac{\mathbf{a} \times \mathbf{p}}{mc^2(\gamma + 1)}, \quad \boldsymbol{\Omega}^{(2)} = -\boldsymbol{\omega}, \quad \gamma = \frac{\sqrt{m^2 c^4 + c^2 \mathbf{p}^2}}{mc^2}. \end{aligned} \quad (74)$$

Here $\mathbf{l} = \mathbf{r} \times \mathbf{p}$ is the angular momentum operator. There are no any terms of order of $\hbar^2$ nonvanishing in both the nonrelativistic and the weak field approximations. We also emphasize the absence of the spin-dependent Mathisson force. Let us stress that Eq. (74) is derived for the strong kinematical effects when the ratios $|\mathbf{a} \cdot \mathbf{r}|/c^2$ and $|\boldsymbol{\omega} \times \mathbf{r}|/c$ are not small. Nevertheless, Hamiltonian (74) coincides in the case of $\boldsymbol{\omega} = 0$ with the

result [25] obtained for the special case of the accelerated frame in the weak field approximation $|\mathbf{a} \cdot \mathbf{r}|/c^2 \ll 1$. When $\mathbf{a} = 0$, we reproduce the exact FW Hamiltonian deduced in Ref. [34] for Dirac particles in the rotating frame. Eq. (74) and the corresponding Hamiltonian obtained in Ref. [34] generalize the approximate expression derived by Hehl and Ni [13].

The equivalent form of $\mathcal{H}_0$ reads

$$\mathcal{H}_0 = \beta (mc^2 + m\mathbf{a} \cdot \mathbf{r}) \gamma - \frac{i\hbar\beta \mathbf{a} \cdot \mathbf{p}}{2mc^2 \gamma} - \boldsymbol{\omega} \cdot \mathbf{l}. \quad (75)$$

The metric under consideration is non-Minkowskian at any parts of finite and infinite trajectories of particles. Therefore, we need to use the contravariant 4-velocity U^i or the world velocity $\mathbf{V} = c\mathbf{U}/U^0$. Evidently, $\mathbf{V}/V = \mathbf{U}/U$. The world velocity operator is given by

$$\mathbf{V} = \frac{d\mathbf{r}}{dt} = \frac{i}{\hbar} [\mathcal{H}_{FW}, \mathbf{r}] = \frac{\beta}{2} \left\{ \left(1 + \frac{\mathbf{a} \cdot \mathbf{r}}{c^2} \right), \frac{c\mathbf{p}}{\sqrt{m^2c^2 + \mathbf{p}^2}} \right\} - \boldsymbol{\omega} \times \mathbf{r}. \quad (76)$$

As a result, the semiclassical approximation for the world acceleration operator

$$\mathbf{w} = \frac{d\mathbf{V}}{dt} = \frac{i}{\hbar} [\mathcal{H}_{FW}, \mathbf{V}]$$

reads [15]

$$\mathbf{w} = -\mathbf{a} \left(1 + \frac{\mathbf{a} \cdot \mathbf{r}}{c^2} \right) - \boldsymbol{\omega} \times (2\mathbf{V} + \boldsymbol{\omega} \times \mathbf{r}) + \frac{(\mathbf{a} \cdot (2\mathbf{V} + \boldsymbol{\omega} \times \mathbf{r}))}{c^2 + \mathbf{a} \cdot \mathbf{r}} (\mathbf{V} + \boldsymbol{\omega} \times \mathbf{r}). \quad (77)$$

The exact semiclassical Eq. (77) agrees with the particular results obtained in Refs. [25] and [34] (which are approximate for the acceleration $\mathbf{a}$).

The expression for the angular velocity of rotation of the vector $\mathbf{N} = \mathbf{V}/V$ is similar to Eq. (63):

$$\boldsymbol{\omega}_V = \frac{1}{V} \mathbf{N} \times \frac{d\mathbf{V}}{dt}. \quad (78)$$

Explicitly,

$$\begin{aligned} \boldsymbol{\omega}_V = & -2\boldsymbol{\omega} + \frac{\mathbf{N}}{V} \times \left[-\mathbf{a} \left(1 + \frac{\mathbf{a} \cdot \mathbf{r}}{c^2} \right) - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \right. \\ & \left. + \frac{(\mathbf{a} \cdot (2\mathbf{V} + \boldsymbol{\omega} \times \mathbf{r}))}{c^2 + \mathbf{a} \cdot \mathbf{r}} (\boldsymbol{\omega} \times \mathbf{r}) \right]. \end{aligned} \quad (79)$$

The exact semiclassical expression for the angular velocity of spin rotation can be written in terms of $\mathbf{v}$ [15]:

$$\boldsymbol{\Omega} = \frac{\mathbf{a} \times (\mathbf{v} + \boldsymbol{\omega} \times \mathbf{r})}{c^2 \left[1 + \frac{\mathbf{a} \cdot \mathbf{r}}{c^2} + \sqrt{\left(1 + \frac{\mathbf{a} \cdot \mathbf{r}}{c^2} \right)^2 - \left(\frac{\mathbf{v} + \boldsymbol{\omega} \times \mathbf{r}}{c} \right)^2} \right]} - \boldsymbol{\omega}. \quad (80)$$

To consider the helicity evolution, it is sufficient to examine three specific cases. If $|\boldsymbol{\omega} \times \mathbf{r}| \ll c$, $|\mathbf{a} \cdot \mathbf{r}| \ll c^2$, one may retain only the terms linear in $\boldsymbol{\omega}$, $\mathbf{a}$:

$$\boldsymbol{o}_v = \boldsymbol{\Omega} - \boldsymbol{\omega}_v = \boldsymbol{\omega} + \frac{\mathbf{v} \times \mathbf{a}}{\gamma v^2}. \quad (81)$$

When $\mathbf{a} = 0$ (a purely rotating frame), we find

$$\boldsymbol{o}_v = \boldsymbol{\omega} + \frac{\mathbf{v} \times (\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}))}{v^2}. \quad (82)$$

The presented solution for the rotating frame is exact.

When $\boldsymbol{\omega} = 0$ (an uniformly accelerated frame), we obtain

$$\boldsymbol{o}_v = \frac{\mathbf{v} \times \mathbf{a}}{v^2} \left(1 + \frac{\mathbf{a} \cdot \mathbf{r}}{c^2} \right) - \frac{\mathbf{v} \times \mathbf{a}}{c^2 \left[1 + \frac{\mathbf{a} \cdot \mathbf{r}}{c^2} + \sqrt{\left(1 + \frac{\mathbf{a} \cdot \mathbf{r}}{c^2} \right)^2 - \frac{v^2}{c^2}} \right]}. \quad (83)$$

As follows from the Eq. (76), $v/c = 1 + \mathbf{a} \cdot \mathbf{r}/c^2$ for the ultrarelativistic particles with negligible masses. For such particles, we find $\boldsymbol{o}_v = 0$. Therefore, their helicity remains unchanged in the uniformly accelerated frame.

We can conclude that the motion in the general noninertial (arbitrarily rotating and accelerating) frame leads to the change of the helicity even for the ultrarelativistic particle with a negligible mass. A similar effect takes place in a gravitational field when a particle trajectory is finite.

Another possible application of our general results is a rotating massive thin shell which metric obtained by Brill and Cohen [35] has the form (10),(25). We will analyse this elsewhere.

7 Squared Dirac equation and the equivalence principle

The electromagnetic and gravitational contributions to the covariant derivative (17) manifest an obvious similarity of the electromagnetic and gravitational effects. In this section we further clarify this similarity by analysing the squared Dirac equation.

The commutator of the covariant derivative (17) reads

$$D_i D_j - D_j D_i = \frac{iq}{\hbar c} F_{ij} + \frac{i}{4} \sigma_{\alpha\beta} R_{ij}{}^{\alpha\beta}. \quad (84)$$

Here $F_{ij} = \partial_i A_j - \partial_j A_i$ is the electromagnetic field tensor, and $R_{ij}{}^{\alpha\beta}$ is the Riemann curvature tensor.

Acting with the conjugate Dirac operator $(i\hbar\gamma^\alpha D_\alpha + mc)$ on (16), we find the squared Dirac equation

$$(-\hbar^2 g^{ij} D_i D_j - \hbar^2 \gamma^{[i} \gamma^{j]} D_i D_j - m^2 c^2) \psi = 0. \quad (85)$$

Substituting Eq. (84), we find

$$\left(-\hbar^2 g^{ij} D_i D_j - \frac{i\hbar q}{2c} \gamma^{[i} \gamma^{j]} F_{ij} + \frac{\hbar^2}{8} \gamma^{[i} \gamma^{j]} \gamma_{[\alpha} \gamma_{\beta]} R_{ij}{}^{\alpha\beta} - m^2 c^2 \right) \psi = 0. \quad (86)$$

Expanding the product of gamma matrices, we derive the equation

$$\left(-\hbar^2 g^{ij} D_i D_j - \frac{\hbar q}{2c} \sigma^{\alpha\beta} F_{\alpha\beta} + \frac{\hbar^2}{4} R - m^2 c^2 \right) \psi = 0, \quad (87)$$

where $F_{\alpha\beta} = e_\alpha^i e_\beta^j F_{ij}$ are the tensor-like electromagnetic field coefficients.

The special form of Eq. (87) for the Dirac particle in a gravitational field has been obtained in Ref. [36].

Explicit computation yields

$$-\hbar^2 g^{ij} D_i D_j = \pi^i \pi_i - \frac{\hbar}{4} \sigma^{\alpha\beta} \{ \pi^i, \Gamma_{i\alpha\beta} \} + \frac{\hbar^2}{16} \sigma^{\alpha\beta} \sigma^{\mu\nu} \Gamma_{\alpha\beta}^i \Gamma_{i\mu\nu}. \quad (88)$$

Since

$$\sigma^{\alpha\beta} \sigma^{\mu\nu} \Gamma_{\alpha\beta}^i \Gamma_{i\mu\nu} = \frac{1}{2} \{ \sigma^{\alpha\beta}, \sigma^{\mu\nu} \} \Gamma_{\alpha\beta}^i \Gamma_{i\mu\nu} = 2 \Gamma_{\alpha\beta}^i \Gamma_i{}^{\alpha\beta} + i \varepsilon^{\alpha\beta\mu\nu} \Gamma_{\alpha\beta}^i \Gamma_{i\mu\nu} \gamma_5,$$

we get finally [15]

$$\left[\pi^i \pi_i - \frac{\hbar}{2} \sigma^{\alpha\beta} \left(\frac{q}{c} F_{\alpha\beta} + m \Phi_{\alpha\beta} \right) + \frac{\hbar^2}{4} R + \frac{\hbar^2}{16} (2\Gamma^i_{\alpha\beta} \Gamma_i^{\alpha\beta} + i\varepsilon^{\alpha\beta\mu\nu} \Gamma^i_{\alpha\beta} \Gamma_{i\mu\nu} \gamma_5) - m^2 c^2 \right] \psi = 0, \quad (89)$$

where

$$\Phi_{\alpha\beta} = \frac{1}{2m} \{ \pi^i, \Gamma_{i\alpha\beta} \}, \quad \gamma_5 = -i\hat{\gamma}^0 \hat{\gamma}^1 \hat{\gamma}^2 \hat{\gamma}^3. \quad (90)$$

In the semiclassical approximation, $\pi^i = mU^i$, and $\Phi_{\alpha\beta}$ coincides with the spin (and momentum) transport matrix in a gravitational field (see Ref. [19]) and with the tensor-like coefficients $\gamma_{\alpha\beta\lambda} u^\lambda$ [27]. It is analogous to the electromagnetic field tensor and leads to the Dirac gyro-gravitomagnetic ratio $g_{grav} = 2$ in perfect agreement with the equivalence principle which is also manifested in the interaction of spin with gravity [37]. This means [33] the absence of both the anomalous gravitomagnetic moment and the gravitoelectric dipole moment which are gravitational analogs of the anomalous magnetic moment and the electric dipole moment, respectively.

Eq. (89) explicitly shows a similarity of the Dirac particle interactions with electromagnetic and gravitational fields [15]. This similarity is caused by the similarity of the motion of spinning particles in any external classical fields shown in Ref. [19].

8 Conclusions

The use of new mathematical methods in relativistic quantum mechanics has allowed us to take an extraordinarily big step in the quantum-mechanical description of dynamics of a Dirac particle in arbitrarily strong gravitational fields as well as in noninertial frames. The derivation of the Hermitian Dirac Hamiltonian (24) has presented the initial Dirac equation for a fermion in arbitrary electromagnetic and gravitational fields in the Schrödinger form. The use of the Schwinger gauge is an important part of this derivation. After this, the relativistic FW transformation has led to the

FW Hamiltonian describing the relativistic particle in strong gravitational fields. This has allowed us to obtain the general quantum-mechanical equations of motion and their classical limit. Other methods of solution of this problem are much less effective.

We have also derived the classical equation (57) for the spin precession in the general gravitational field (in the Schwinger gauge) and have observed its consistency with the classical limit of the corresponding quantum-mechanical equation. This conclusion has been confirmed by deriving and analyzing the squared Dirac equation. The general quantum-mechanical and classical equations for the evolution of the helicity have been obtained. Their application to the dynamics in the general noninertial frame has revealed the change of the helicity even for the ultrarelativistic particle with a negligible mass.

The similarity of the quantum-mechanical equations derived in the FW representation and the corresponding classical ones opens a possibility of unification of the relativistic quantum mechanics. A possibility of such an unification with the use of the above representation follows also from other results [38, 39, 40].

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